On Euler's Summability

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Knopp has proved the following theorems on introducing Euler's summability. In this paper I will briefly prove them by using Anonholdclebsch notation.

Let p be 0 or a positive integer, put $w=E(z)=\frac{z}{q+1-qz}(q=2^p-1)$, and then in the series $\sum_{n=0}^{\infty} a_n$, put $\sum_{n=0}^{\infty} a_n w^{n+1}=\sum_{n=0}^{\infty} a_n{}^{(p)} z^{n+1}$ formally. Then we get the following theorems.

Theorem I.

$$a_n{}^{(p)}=\frac{1}{(q+1)^{n+1}}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}a_\nu \tag{1}$$

$$a_n{}^{(p+p')}=\frac{1}{(q+1)^{n+1}}\sum_{\nu=0}^{n}\binom{n}{\nu}q^{n-\nu}a_\nu^{(p')} \tag{2}$$

where p' is 0 or a positive integer.

Theorem II.

$$\frac{a_n}{q+1}=\sum(-1)^\nu\binom{n}{\nu}(q+1)^{n-\nu}a_{n-\nu}{}^{(p)}q^\nu$$

Theorem III.

$$a_0{}^{(p)}+a_1{}^{(p)}+\cdots\cdots+a_{n-1}{}^{(p)}=\frac{1}{(q+1)^n}\sum\binom{n}{\nu}q^{n-\nu}S_\nu$$

where $S_0=0,\quad S_\nu=a_0+a_1+\cdots\cdots+a_{\nu-1}$

Theorem III will be shown as below,

$$\sum a_n w^{n+1}=\sum a_n{}^{(p)}z^{n+1}$$

$$\frac{1}{1-z}=\frac{1+qw}{1-w}$$

therefore

$$(1+qw)\frac{\sum a_n w^{n+1}}{1-w}=\frac{\sum a_n{}^{(p)}z^{n+1}}{1-z}$$

$$\therefore \quad (1+qw)\sum_{n=0}^{\infty} s_n w^n = \sum_{n=1}^{\infty} s_n{}^{(p)} z^n \tag{1}$$

where

$$a_0{}^{(p)}+a_1{}^{(p)}+\cdots\cdots+a_{n-1}{}^{(p)}=s_n{}^{(p)}$$

From (1), we get

$$\sum_{n=0}^{\infty} s_{n+1} w^{n+1}+q\sum_{n=0}^{\infty} s_n w^{n+1}=\sum_{n=1}^{\infty} s_n{}^{(p)} z^n \tag{2}$$

If we substitute $w=E(z)$ in the left hand side of (2), and calculate the coefficients of z^n by using Theorem I, we get

$$\frac{1}{(q+1)^n}\left\{\sum_{\nu=0}^{n-1}\binom{n-1}{\nu} q^{n-1-\nu} s_{\nu+1}+q\sum_{\nu=0}^{n-1}\binom{n-1}{\nu} q^{n-1-\nu} s_\nu\right\}$$

$$=\frac{1}{(q+1)^n}\sum_{\nu=0}^{n}\binom{n}{\nu} q^{n-\nu} s_\nu$$

In this paper I will prove the above theorems by using Anonhold-clebsch notation.

Proof of theorem 〔I〕

(1) $$\sum a_n w^{n+1}=\sum_{n=0}^{\infty} a^n w^{n+1}=\frac{w}{1-aw}\quad\frac{w}{\dfrac{1}{w}-a}=\frac{1}{\dfrac{q+1}{z}-q-a}$$

$$=\frac{1}{(q+1)-(q+a)z}=\frac{z}{q+1}\quad\frac{1}{1-\dfrac{q+a}{q+1}z}$$

$$=\sum\frac{z}{q+1}\left(\frac{q+a}{q+1}\right)^n z^n$$

Therefore we get

$$\overset{(p)}{a_n}=\frac{(q+a)^n}{(q+1)^{n+1}}=\frac{1}{(q+1)^{n+1}}\sum_{\nu=0}^{n}\binom{n}{\nu} q^{n-\nu} a_\nu=\frac{1}{q+1}\left(\frac{q+a}{q+1}\right)^n$$

(2) $$\frac{1}{(q+1)^{n+1}}\sum_{\nu=0}^{n}\binom{n}{\nu} q^{n-\nu} a_\nu\,(p')$$

$$=\frac{1}{q+1}\left(\frac{q+a(p')}{q+1}\right)^n=\frac{1}{(q+1)(q'+1)}\frac{\left(q+\frac{q'+a}{q'+1}\right)^n}{(q+1)^n}$$

$$\frac{1}{q''+1}\left(\frac{q''+a}{q''+1}\right)^n=a_n^{(p+p')}$$

where $q'=2^{p'}-1,\quad q''=2^{p+p'}-1$

Proof of theorem II.

Theorem II will be rewritten as follows,

$$\frac{a_n}{q+1}=\left\{(q+1)a^{(p)}-q\right\}^n$$

By theorem I,

$$\frac{a_n}{q+1}=\left\{(q+1)\frac{q+a}{q+1}-q\right\}^n=\frac{a^n}{q+1}=\frac{a_n}{q+1}$$

Proof of theorem III.

$$S_n^{(p)}=\sum_{i=0}^{n-1}a^{(p)i}=\sum_{i=0}^{n-1}\frac{(q+a)^i}{(q+1)^{i+1}}=\frac{1}{q+1}\frac{1-(q+a/q+1)^n}{1-\frac{q+a}{q+1}}$$

$$=\frac{1}{(q+1)^n}\frac{(q+1)^n-(q+a)^n}{1-a}=\frac{1}{(q+1)^n}\sum\binom{n}{\nu}\frac{q^{n-\nu}(1-a)^\nu}{1-a}$$

$$=\frac{1}{(q+1)^n}\sum\binom{n}{\nu}q^{n-\nu}(a_0+a_1+\cdots\cdots+a^{\nu-1})$$

Reference

(1) Knopp : Math. Zeits. 18 (1929)

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