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HARRODIAN INSTABILITY AND SMOOTH FACTOR SUBSTITUTION [*]

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Reflecting on the history of macrodynamics since Sir Roy Harrod's *An Essay in Dynamic Economics* (1939) was published, one may notice that, while the relation between the natural and warranted rates of growth, two of the key concepts of Harrod's dynamic economics, has drawn considerable attention in the vast literature, the other relation, between the actual and warranted rates of growth which Harrod discussed under the name of the *Instability Principle,* has not been widely examined, although there have appeared a few remarkable contributions[1] clarifying the issue.

Presumably, it is the case that this unpopularity of the Instability Principle comes, at least partly, from a customary view which has, as Nagatani (1981, pp. 4 − 5) remarks, prevailed since Solow's monumental work (1956). It is that the Harrodian "Knife-edge" depends on the extreme assumption that there is no factor substitution.

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(1) For example, Alexander (1950), Rose (1959), Nelson (1961), Okishio (1964), Nikaido (1975, 80) and Adachi (1982).

This view seems unfair to Harrod; he never stated it。 On the contrary, in some places he explicitly argued that the Instability Principle holds even under the condition of smooth factor substitution. For example, in his "Comment" on Philvin (1953), and in *Economic Dynamics* (1973, pp. 44−5), Harrod clearly took into account the influences of the change in the rate of interest on the normal capital-output ratio through factor substitution and argued that this influence was not enough for ensuring stability of the warranted growth path even under the condition of smooth factor substitution.

However, it also seems to us that Harrod left incomplete the analysis concerning the implications of smooth factor substitution for the Instability Principle. For, he did not examine the possibility that the change in other factor prices, particularly the real wage rate, influences the normal capital-output ratio through factor substitution.

The reasons which led Harrod to reject the stabilizing effects of the rate of interest may not be applicable to the real wage rate; first, the decline in the real wage rate caused by an upward deviation of the actual from the warranted rate of growth may not be regarded as temporary, even if, as Harrod (1973, p.45) claimed, so is the rise in the rate of interest due to the upward deviation; and second, the wage is, in general, a much more important element in cost than interest.

Therefore, it is an open question whether and, if so, under what conditions Harrodian instability, can hold when the effects of the change in the real wage rate on the normal capital-output ratio through factor substitution are taken into account. In the present paper we aim to develop a fairly simple macrodynamic model of Harrodian type and to examine this question.

Nikaido (1975, 1980) developed a variant of neoclassical growth models which incorporates smooth factor substitution as well as a Harrodian investment function and argued that smooth factor substitution cannot remove Harrodian instability. However, his argument depends on the assumption that the prices of goods are exogeneous at each point of time: If the prices are momentarily flexible, then the discrepancy between the actual and warranted rates of growth would not occur in his model.

It seems rather restrictive to construe price-rigidity in goods markets as indispensable for Harrodian instability. We shall accomplish our analytical purpose under the assumption that the prices of goods are so flexible that demand and supply in goods markets are always equal with each other.

To our knowledge, Okishio (1964) is the only preceeding work dealing with Harrodian instability under the same assumptions as ours, i. e., smooth factor substitution and price-flexibility in goods markets. Thus, subsequently we shall relate our model to his.

The present paper is structured as follows. Section I sets out our assumptions and develops a Harrodian model. Section II analyzes the model and proves the main proposition of the present paper with respect to Harrodian instability. Section II also discusses Okishio's model in order to make clear the difference in conclusion between his and ourselves. Section III contains some concluding remarks concerning our results.

I. THE ASSUMPTIONS AND THE MODEL

First, it is assumed, in common with the standard neoclassical one-sector model, that one single all-purposed good Y is produced with the aid of capital K and labour N. We shall, however deviate from the doctorine with respect to the extent of capital-malleability: We assume that the representative entrepreneur must incur some cost in order to transform his existing capital stock from embodying one level of factor intensity to embodying another. In the following, we shall refer to this cost as *transformation cost.*

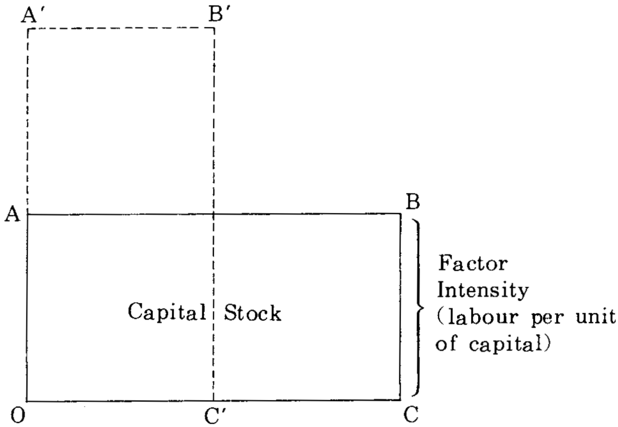

Fig. 1 Capital stock embodying a level of factor intensity.

An existing capital stock embodying a level of factor intensity x^* $\equiv$ (labour)/(capital) can be symbolically visualized by a rectangule ABCO in FIG. 1 where the length of the segment OA and the area of ABCO represent $1/x^*$ and K, respectively. One can imagine a factory in which K units of capital equipment are divided into x^*K groups and those groups are appropriately placed so that, if each worker is equipped with $1/x^*$ units of capital equipments, then he can be most efficiently engaged in production. Making use of Fig. 1, the second assumption stated in the previous paragraph may be restated as follows: It is costly for the representative entrepreneur to change the shape of the rectangul while keeping its area intact: say from ABCO to A'B'C'O.

The third assumption is that for any given K and N the output Y is technologically maximized when the capital stock embodies the level of factor intensity equal to N/K. The fourth assumption is that the technological relation between the inputs, K embodying x^* and N, and the output Y can be expressed as

$$Y = F(K, N, x^*) \qquad (1),$$

which is supposed to be linearly homogeneous with respect to K and N and to have positive and diminishing partial derivatives for each of them, but not necessarily for x^*. Thus, we can rewrite (1) as

$$Y/K = f(x, x^*), \qquad x \equiv N/K \qquad (2),$$

where for any $x > 0$

$$\begin{aligned} f_1(x, x^*) &\equiv \frac{\partial}{\partial x} f(x, x^*) > 0 \\ f_{11}(x, x^*) &\equiv \frac{\partial^2}{\partial x^2} f(x, x^*) < 0 \end{aligned} \qquad (3).$$

In terms of $f(x, x^*)$, the third assumption can be restated as follows. For any positive x and x^*,

$$G(x^*) \equiv f(x^*, x^*) > f(x, x^*), \text{ if } x \neq x^* \qquad (4).$$

Based on (3) and (4), one can easily verify that when x is sufficiently close to x^*,

$$G'(x^*) \equiv \frac{d}{dx^*} G(x^*) = f_1(x^*, x^*) \gtreqless f_1(x, x^*), \quad \text{according as } x \gtreqless x^* \qquad (5)$$

$$G''(x^*) \equiv \frac{d^2}{dx^{*2}} G(x^*) = f_{11}(x^*, x^*) + f_{12}(x^*, x^*)$$
$$f_2(x, x^*) \equiv \frac{\partial}{\partial x^*} f(x, x^*) \gtreqless 0, \quad \text{according as} \quad x \gtreqless x^* \qquad (5),$$

where $f_{12}(x, x^*) \equiv \frac{\partial^2}{\partial x \partial x^*} f(x, x^*)$. In what follows, we assume $G''(x)$ is negative for any positive x^*(the law of marginal productivity). The graphs of $f(x, x^*)$ and $G(x^*)$ are depicted in Fig. 2.(2)

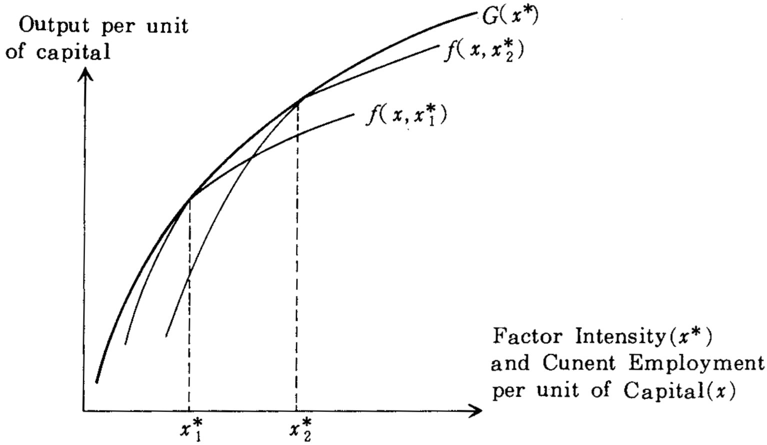

Fig. 2 The graphs of $G(x^*)$ and $f(x, x^*)$

Next, let us state the assumptions concerning the behaviour of the representative entrepreneur. Let R be the current real wage rate and let R^* be the rate which the representative entrepreneur expects to prevail on average over his time-horizon, where the variance of R^* is supposed to be sufficiently small. Now, if the existing capital stock were per-

(2) The explicit distinction between $f(x, x^*)$ and $G(x^*)$ such as shown in Fig. 2 was, for the first time, obtained by Okishio (1984), although the reasoning which made him arrive at this figure is quite different from ours.

fectly malleable, it would be best for the representative entrepreneur to choose the level of factor intensity satisfying $R^{*} = G'(x^{*})$ at each point of time. However, under the second assumption stated at the beginning of the present section, such a choice means that the representative entrepreneur must make up his mind to incur some transformation cost mentioned there. If that cost is very great, it might be better to choose the level of factor intensity satisfying

$$R^{*} = G'(x^{*}) \qquad (6),$$

which implies that the representative entrepreneur seeks to maximize average expected real profit per unit of capital, $\pi^{*} \equiv G(x^{*}) - R^{*} x^{*}$.

In what follows, that is assumed to be the case. Thus, the fifth assumption is that at each point of time the capital stock embodies the level of factor intensity satisfying (6). That is, the representative entrepreneur adjusts the shape of the rectangul illustrated in Fig. 1 to the gradual revision of R^{*} so that (6) can hold at each point of time.

With respect to this revison of R^{*}, we shall employ the adaptive expectation hypothesis;

$$\dot{R}^{*} \equiv \frac{d}{dt} R^{*} = a(R - R^{*}), \qquad a > 0 \qquad (7),$$

which makes the sixth assumption.

Under predetermined x^{*} and K, and currently prevailing R, the representative entrepreneur makes a decision on the level of current production. It is the seventh assumption that the representative entrepreneur intends to maximize current real profit per unit of capital, $\pi \equiv f(x, x^{*}) - Rx$, by choosing an appropriate x. Thus, we have

$$R = f_1(x, x^{*}) \qquad (8)$$

The remaining important decision to be made by the representative entrepreneur is on the rate of capital accumulation, $g \equiv \dot{K}/K$. The literature referred to in footnote 1 generally assumes that the investment behaviour of the representative entrepreneur raises (reduces) g if the actual output per unit of capital falls short of (exceeds) some normal output per unit of capital. In our model this investment behaviour is justified as follows: Suppose that $f(x, x^*) > G(x^*)$. It follows, considering (6) and (8), that π is greater than π^*. Therefore, the representative entrepreneur may judge that it is to his advantage to raise g now, since by so doing he can earn more than the normal level of profit in the near future. By symmetrical reason, if $f(x, x^*) < G(x^*)$, then the representative entrepreneur may judge that it is to his advantage to reduce g until R^* becomes actualized.

Based on the foregoing argument, the investment behaviour of the representative entrepreneur can be assumed to be as follows,

$$\dot{g} = b(f(x, x^*) - G(x^*)), \qquad b > 0 \qquad (9)^{(3)},$$

which makes the eighth assumption.

Last, let us state the ninth assumption concerning the goods market. It is that the price of the good is flexible enough to clear the goods market at each point of time. Hence, we have

$$g = s^d f(x, x^*) \qquad (10),$$

(3) Strictly speaking, $f(x, x^*) = G(x^*)$ does not imply that the representative entrepreneur leaves the rate of capital accumulation g intact; for example, he might leave intact the level of investment gK. This point was originally raised by Alexander (1950). Thus the ninth assumption should be qualified to allow the possibility that, whenever the representative entrepreneur expects the current situation to continue in the long-run, he leaves the rate of capital accumulation g intact.

where it is assumed, for simplicity, that the desired propensity to save, s^d, is positive, constant and less than unity.

The equations, (6)−(10), compose our Harrodian dynamic model. It should be noted that throughout our argument the volume of employment is supposed to be always smaller than the workforce.

II. THE MAIN PROPOSITION

Now, let us obtain our main proposition By simple calculation, one can verify that our dynamic model is expressed by the system of differential equations,

$$\dot{x} = \frac{b(f(x,x^*) - G(x^*))}{s^d f_1(x,x^*)} - \frac{f_2(x,x^*)a(f_1(x,x^*) - G'(x^*))}{G''(x^*)f_1(x,x^*)}$$

$$\dot{x}^* = \frac{a(f_1(x,x^*) - G'(x^*))}{G''(x^*)} \qquad (11).$$

It should be noted that $\dot{x} = \dot{x}^* = 0$ whenever $x = x^*$.

How, then, may an integral curve of this system be depicted in a neighbourhood of the straight line $x = x^*$? To see this, consider the slope of an integral curve of the system,

$$\frac{dx}{dx^*} = \frac{\dot{x}}{\dot{x}^*} = \frac{G''(x^*)b(f(x,x^*) - G(x^*))}{s^d a(f_1(x,x^*) - G'(x^*))} - \frac{f_2(x,x^*)}{f_1(x,x^*)} \qquad (12).$$

Let (x, x^*) converge to any point on the straight line $x = x^*$ along an integral curve crossing the point. One can derive(4)

$$\lim_{x^* \to x} \frac{dx}{dx^*} = \frac{bG''(x)}{s^d a f_{11}(x,x^*)} \qquad (13).$$

(4) See APPENDIX.

Thus, if (13) is greater (smaller) than unity on the straight line $x=x^*$, then any integral curve of the system (11) can be drawn as in Fig. 3 (Fig. 4): The system (11) is unstable (stable) in the sense that, once there happens to occur a discrepancy between x and x^*, the discrepancy gets wider (narrower), as time passes.

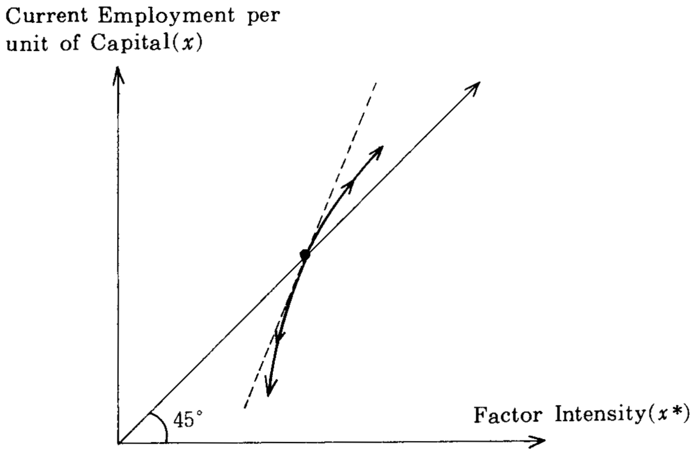

Fig.3 An integral curve (unstable case)

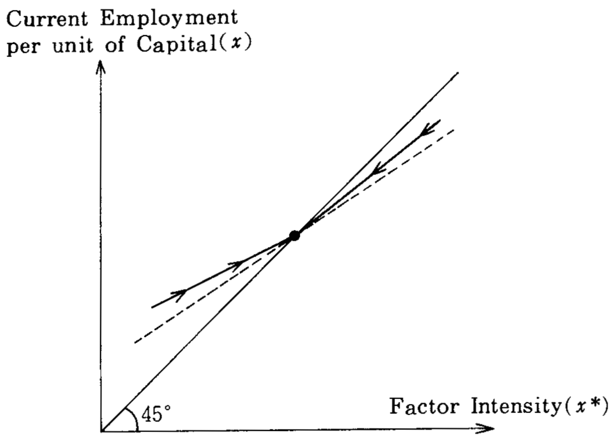

Fig.4 An integral curve (stable case)

Now, define $g_w \equiv s^d G(x^*)$. If $g = g_w$, then $R = R^*$ and $x = x^*$. It means that the representative entrepreneur is making the best decisions both on the level of current production and on the level of factor intensity under current and expected real wage rates and that under the investment function formulated as (9) he has no incentive to change the level of g. Hence, g_w may be regarded as virtually equivalent to Harrod's warranted rate of growth(5), except that the former is the rate of growth *of capital stock* while the latter is *of output*. In what follows, we shall call g_w the warranted rate of growth(6). Making use of those terms, the main proposition is stated as follows.

PROPOSITION: The equilibrium growth path on which there is no discrepancy between the actual and warranted rates of growth is unstable (stable) in the sense that, once a discrepancy between those rates of growth happens to occur, it monotonically gets wider (narrower) as time passes, *if (13) is greater (smaller) than unity.*

Proof: Logarithmically differentiating g_w/g with time, we see

$$\frac{\dot{g}_w}{g_w} - \frac{\dot{g}}{g} = \frac{a(f_1(x,x^*) - G'(x^*))}{G''(x^*)} - \frac{b(f(x,x^*) - G(x^*))}{s^d f(x,x^*)}$$

$$\gtreqless 0, \text{ according as } x \lesseqgtr x^* \tag{14}$$

and the proposition follows.

Suppose that R is smaller than R^*. Then, the level of current production is above normal, i. e., $x > x^*$, and the representative en-

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(5) See Harrod (1939, p. 16).

(6) $g = s^d f(x, x^*)$ and $g_w = s^d G(x^*)$ correspond to Harrod's fundamental equations, $G = s^d/C$ and $G_w = s^d/C_r$, respectively.

trepreneur raises the rate of capital accumulation, which induces, through the multiplier effect, $f(x, x^*)$ to increase and R to decline. Since, as is clear from (7), this decline brings about the decline in R^*, the representative entrepreneur comes to choose a more labour intensive technology of production, i. e., a higher x^*. As a result, $G(x^*)$ must increase. Considering the investment function (9), it may be seen that the increase in $f(x, x^*)$ makes g greater while the increase in $G(x^*)$ makes g smaller. Hence, if the former effect dominates the latter, then Harrodian instability arises.

From the foregoing argument, one can easily verify how each component of (13) may be related to Harrodian instability. For example, the greater the income-multiplier, $1/s^d$, the more rapidly $f(x, x^*)$ increases, which obviously implies that Harrodian instability is more probable. Other parameters (a, b, f_{11}, G'') could be checked similarly.

As mentioned earlier, Okishio (1964) developed a Harrodian macrodynamic model and claimed that Harrodian instability is *always* established even under the assumptions of smooth factor substitution and price-flexibility in goods markets, assumption which we also employ. The difference in conclusion between Okishio and ourselves comes from his other assumption that the rate of capacity utilization, which is roughly comparable with $f(x, x^*)/G(x^*)$, is an increasing function of current real profit per unit of capital; $u(\pi)$ and $u'(\pi)>0$.

Moreover, it is also assumed that the investment function of the representative entrpreneur is

$$\dot{g}=b(u(\pi)-1), \qquad b>0 \qquad (15).$$

For simplicity, suppose workers do not save. Denoting by s_π the propensity to save of 'capitalists', we have $g=s_\pi\pi$ as the market clearing condition with respect to the goods market. Substituting it to (15), we see

$$\dot{g} = b(u(g/s_\pi) - 1) \quad ,$$

which is always unstable if s_π is positve. That is the gist of Okishio's argument.

It does not seem to us that smooth factor subsitution plays any substantial role in his argument. Making reference to his other works(7), it is, as it were, the short-term version of Keynes' "animal spirits" that is supposed to underlie the relation between u and π, and it is rather unclear how this relation is affected by smooth factor substitution.

Let us consider his argument in the light of our model. Since π is equal to $f(x, x^*) - Rx$, considering (3) and (7), one can verify that $f(x, x^*)/G(x^*)$ is an increasing function of π. Furthermore, it also depends on normal real profit per unit of capital, π^*, through x^* in $f(x, x^*)$. Hence, (15) can be rewritten to

$$\dot{g} = b(u(\pi, \pi^*) - 1) \quad ,$$

which is virtually equivalent to (9). If Okishio's argument can be interpreted in that way, then his model is regarded as a polar case of ours in which u is independent of π^* and or in which π^* is constant.

Ⅲ. CONCLUDING REMARKS

We have shown that it depends on the values of the income multiplier $(1/s^d)$, the 'accelerator' (b), the expectation coefficient (u) and the technological parameters (f_{11}, G'') whether or not Harrodian insta-

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(7) Okisio (1976, 1977, 1978).

bility is established under the assumptions smooth factor substitution and price-flexibility in the goods market: Harrodian instability is more probable the greater are $(1/s^d)$ and (b), and or the smaller (a) and f_{11}/G''.

The basic idea of the present paper that the real wage rate is one of the main factors which influences the normal capital-output ratio through factor substitution would not have been rejected by Harrod. In fact, he seems even to have given support to this view by citing the historical fact that "it was the high price of human labour relatively to the prices of coals and other materials that caused the great increase of capital intensity in the U.S.A. in the nineteenth century,……." (Harrod (1973), pp. 68–69)

SUMMARY

A Harrodian dynamic model is constructed, and a condition for Harrodian instability to hold is derived under the assumptions of smooth factor substitution and price-flexibility in goods markets. If the condition is satisfied, then the discrepancy between the actual and warranted rates of growth, once it occurs, monotonically expands over time. According to the condition, the greater the income multiplier and the accelerator and/or the smaller the expectation coefficient concerning the future real wage rate, the more probable is Harrodian instability.

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APPENDIX

Resorting to L'H$\hat{o}$ pitals rule, we see

$$y \equiv \lim_{x^* \to x} \frac{dx}{dx^*} = \frac{G''(x)b\{f_1(x,x)y + f_2(x,x) - G'(x)\}}{s^d f(x,x)a\{f_{11}(x,x)y + f_{12}(x,x) - G''(x)\}} \quad (16).$$

Since $f_2(x, x) = 0$, $f_1(x, x) = G'(x)$ and $f_{11}(x, x) + f_{12}(x, x) = G''(x)$ from (5), (16) is rewritten to

$$y = \frac{bG''(x)(y-1)}{s^d a f_{11}(x,x)(y-1)} \quad (17).$$

Again applying L'H$\hat{o}$ pitals rule to (17), we see

$$1 = \lim_{y \to 1} y = \frac{bG''(x)}{s^d a f_{11}(x,x)} \quad ,$$

which implies (13).

REFERENCES

Adachi, H. (1982). *The Theory of Economic Fluctuation*, (in Japanese) Tokyo, Nikkei.

Alexander, S. S. (1950). "Mr. Harrod's Dynamic Model," *Economic Journal*, vol. 60, p. 724–39.

Harrod, R. (1939). "An Essay in Dynamic Theory," *Economic Journal*, vol. 49, p. 14–33.

——— (1973). *Economic Dynamics*, London: Macmillan.

Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money*, London, Macmillan.

Nagatani, K. (1981). *Macroeconomic Dynamics*, Cambridge, Cambridge University Press.

Nelson, R. R. (1961). "A Note on Stability and the Dehavior Assumptions of Harrod-type Model," *Economic Journal*, vol. 71, p. 335–49.

Nikaido, F. (1975). "Factor Substitution and Harrod's Knife-edge," *Zeitschrift für National ökonomie*, vol. 35, p. 149–54.

——— (1980). "Harrodian Pathology of Neoclassical Growth: the Irrevance of Smooth Factor Substitution," *Zeitschrift für National ökonomie*, vol. 40, p. 111–34.

Okishio, N. (1964). " Instability of Harrod-Domar's Steady Growth," *Kobe University Economic Review*, vol. 10, p. 19–27.

——— (1976). *The Theory of Capital Accumulation*, (in Japanese) Tokyo, Chikuma.

——— (1977). *Modern Economics*, (in Japanese) Tokyo, Chikuma.

——— (1978). *Essays in Modern Economics*, (in Japanese) Tokyo, Chikuma.

——— (1984). "The Discussion of New Investment, Technique and Rate of Utilization," *Kobe University Economic Review*, vol. 30, p. 15–32.

Rose, H. (1959). "The Possibility of Warranted Growth," *Economic Journal*, vol. 69, p. 313–32.

Solow, R. (1956). "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, vol. 52, p. 65–94.