Approach to Determinate Two Constants in the Strahler's Hypsometric Function

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Abstract : The relationship between a area and its altitude in a drainage basin is generally showed Area-Altitude curve.

In 1952, Strahler showed that this curve was expressed by following hypsometric model function, and erosional topography was able to classify into several groups by its specific integral value of the function.

$$y=\left[\frac{1-x}{1+\left(\frac{1-r}{r}\right)x}\right]^{z}$$

in which

x=relative area in a basin, and

y=relative altitude.

In this paper, we expressed two methods to determinate above constants r, z in the function and the results in which r, z values by graphycal method is about 15% smaler than by neumerical method.

Introduction

Area-Altitude curve was proposed by Imamura[1] in 1937, and then Strahler[2] showed model function in regard to the curve in 1952. This function is modified the distribution equation (1) of washload by Rose[3] in 1937.

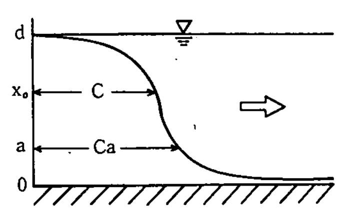

Fig. 1 Density distribution of washload.

$$\frac{C}{C_a}=\left[\frac{d-x_0}{x_0}\ \frac{a}{d-a}\right]^{z_0} \qquad (1)$$

in which

z_0 $=\omega/\kappa u_*$

ω =Karmains constant,

u_* =friction velocity,

d =water depth,

x_0 =distance from bed,

C =density of washload at x, and

C_a =density of washload at x =a.

Hence, $d-a>0$, $r=a/d$, $z\geqq 0$ and let exchange variables as follows

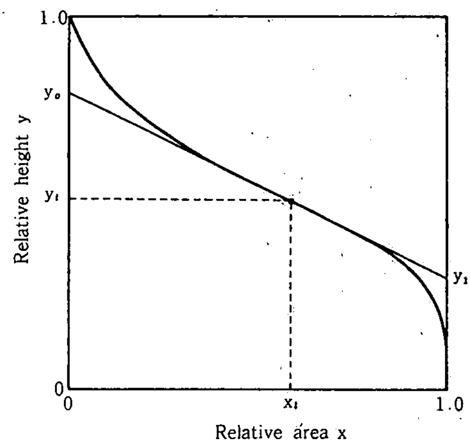

Fig. 2 Illustration of percentage hypsometric curve

$$\frac{C}{C_a}=y=\frac{H_{max}-H}{H_{max}-H_0}, \text{ and } \frac{x_0-a}{d-a}=x=\frac{A}{A_0}.$$

in which

H_{max} = maximum altitude,
H_0 = minimum altitude,
H = altitude,
A = drainage basin upon H, and
A_0 = drainage basin upon H_0.

We get equation (2) well known as Strahler's percentage hypsometric function.

$$y=\left[\frac{r}{1-r}\right]^{z}\cdot\left[\frac{1}{(1-r)x+r}-1\right]^{z} \tag{2}$$

Then, eq. (2) is able to exchange as follows

$$y=\left[\frac{1-x}{1+\left(\frac{1-r}{r}\right)x}\right]^{z} \tag{3}$$

The inflection point of this function is shown coordinates (x_t, y_t) that is expressed in eq. (4) or eq. (5).

$$x_t=\frac{1}{2}\left(\frac{1-2r+z}{1-r}\right)=1-\frac{1}{2}\left(\frac{1-r}{1-z}\right) \tag{4}$$

$$y_t=\left[\frac{r}{1-r}\cdot\frac{1-z}{1+z}\right]^{z}=2(1-x_t)\frac{r}{1-z} \tag{5}$$

And the gradient at the inflection point y_t' is

$$y_t'=-4(1-r)\frac{z}{1-z^2}\left[\frac{r}{1-r}\cdot\frac{1-z}{1+z}\right]^{z}=-\frac{2y_t}{1-x_t}\cdot\frac{z}{1+z} \tag{6}$$

Where we define the gradient coefficient of inflection point

$$I_t = -\frac{y_t'}{y_t} = \frac{2}{(1-x_t)} \quad \frac{z}{1+z} = 4\,(1-r)\frac{z}{1+z^2} \tag{7}$$

Then, the exponent z and constant r, can be determined by eq. (8) and eq. (9) when the inflection point $(x_t,\ y_t)$ and the gradient at this point have been measured.

$$z = -\frac{-y_t'(1-x_t)}{2y_t + y_t'(1-x_t)} = \frac{(1-x_t)I_t}{2-(1-x_t)I_t} \tag{8}$$

$$r = 1 - \frac{1-z}{2(1-x_t)} \tag{9}$$

Neumerical method

The percentage hypsometric function (3) is able to tend Taylor's serieas around $r=r_0$, $z=z_0$ excepting not less second order term.

$$y = f_0 + f_1\Delta r + f_2\Delta z \tag{10}$$

$$\left.\begin{aligned}
f_0 &= y|_{r_0 z_0} & f_{0i} &= \left[\frac{1-x_i}{1+\left(\frac{1-r}{r}\right)x_i}\right]^z \\
f_1 &= \left.\frac{\partial y}{\partial r}\right|_{r_0 z_0} & f_{1i} &= f_{0i}\frac{z}{r^2}\left[\frac{x_i}{1+\left(\frac{1-r}{r}\right)x_i}\right] \\
f_2 &= \left.\frac{\partial y}{\partial z}\right|_{r_0 z_0} & f_{2i} &= f_{0i}\ \ln\left[\frac{1-x}{1+\left(\frac{1-r}{r}\right)x_i}\right]
\end{aligned}\right\} \tag{11}$$

The observation equation is follows on x_i, y_i $(i=1,\ 2,\ \cdots,\ n)$. Where ε_i is residual

$$f_{1i}\Delta r + f_{2i}\Delta z - (y_i - f_{0i}) = \varepsilon_i \tag{12}$$

We can obtain Δr, Δz by adapting the least square method. Next, using the value of $\Delta r + r_0$ or $\Delta z + z_0$ instead of the value of r_0 or z_0, repeating the same method still the condition $|\Delta r/r_0| < \delta$, and $|\Delta z/z_0| \leqq \delta$ are held when δ is allowable error. After several repeating calculation, we get r, z as the most provable value of r_0 or z_0. Let consider the initial value of r_0, z_0. We are taking the following method. When the variable transformation is done as :

$$u = \frac{x}{1-x},\quad v = \frac{1}{y} \tag{13}$$

Eq. (3) change to eq. (14)

$$v^{\frac{1}{z}} = \frac{1+\frac{1-r}{r}x}{1-x} = 1 + \frac{1}{r}u \tag{14}$$

$$u = r\,(v^{\frac{1}{z}} - 1) \tag{15}$$

Now, let derivative of u with respect to v,

$$\frac{du}{dv}=\frac{r}{z}v^{\frac{1}{z}-1} \qquad (16)$$

and take logarithms of both terms,

$$\ln\left(\frac{du}{dv}\right)=\ln\left(\frac{r}{z}\right)+\left(\frac{1}{z}+1\right)\ln(v) \qquad (17)$$

Exchanging the eq. (20) into difference equation with variable transformation as follows,

$$\left.\begin{aligned} &\alpha=\ln\left(\frac{z}{r}\right),\quad \beta=\left(\frac{1}{z}+1\right) \\ &Y_t=\ln\left(\frac{\Delta Y}{\Delta v}\right)=\ln\left(\frac{u_{t+1}-u_t}{v_{t+1}-v_t}\right)=\ln\left[\frac{(x_{t+1}-x_t)}{(1-x_{t+1})(1-x_t)}\cdot\frac{y_t\cdot y_{t+1}}{(y_t-y_{t+1})}\right] \\ &X_t=\frac{1}{2}\ln(v_t\cdot v_{t+1})=-\frac{1}{2}\ln(y_t\cdot y_{t+1}) \end{aligned}\right\} \qquad (18)$$

Then, we obtain the linear equation,

$$Y_t=\alpha+\beta X_t \qquad (19)$$

Using the least square method, we determine the initial value of r, z after simple calculations, as follows,

$$z_0=\frac{1}{\beta-1},\qquad r_0=\frac{z_0}{e^{\alpha}} \qquad (20)$$

Graphycal method

It is necessary to use a computor to determinate r, z by neumerical method. Otherwise after freehand drawing the percentage hypsometric curve based on observation points and looking for the inflection point (x_t, y_t) on the curve and tangential line which gradient

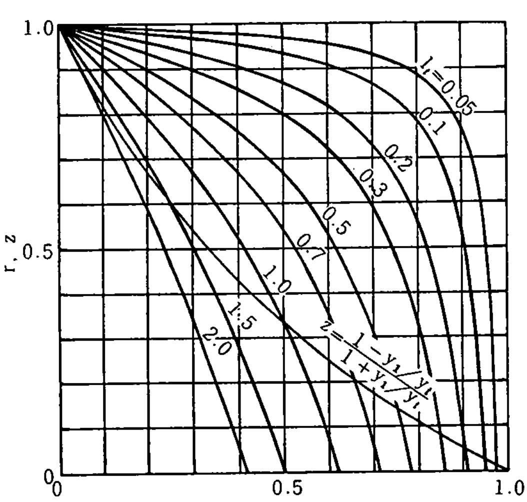

Fig. 3 Relation between r, z and y_1/y_t by eq. (24).

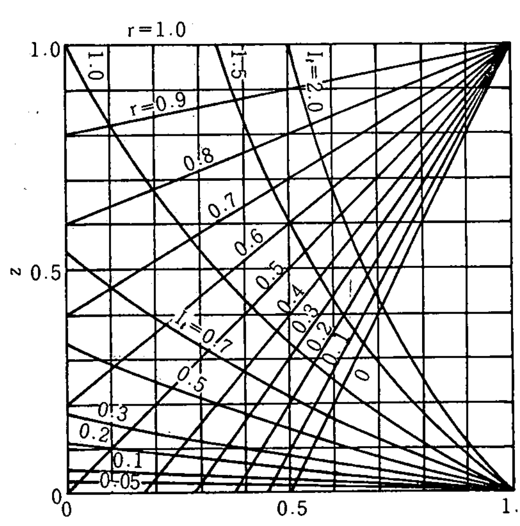

Fig. 4 Relation between z and x_t by eq. (25).

y_t' at the point with the intercept y_0 and y_1 on $x=0$ or $x=1.0$ axis, we can obtain two constants as follows

$$y_t'=y_1-y_0 \tag{21}$$

$$y_1=y_t+y_t'(1-x_t) \tag{22}$$

$$z=\frac{y_t-y_1}{y_t+y_1}=\frac{1-y_1/y_t}{1+y_1/y_t} \tag{23}$$

$$r=\frac{y_t^2-y_1\cdot y_0}{y_t^2-y_1^2}=1-\frac{I_t(y_1/y_t)}{1-(y_1/y_t)^2} \tag{24}$$

But error of r from eq. (24) become great according to be small the defference y_t and y_1 as shown Fig. 3.

Then, from eq. (8) and (9), we obtain Fig. 4.

$$z=\frac{(1-x_t)I_t}{2-(1-x_t)I_t}=1-2(1-r)\ (1-x_t) \tag{25}$$

And from eq. (5) and eq. (7), we obtain Fig. 5.

$$y=\frac{r}{1-r}\cdot\frac{1-z}{1+z}=\frac{(z+4/I_t)z-1}{(z+2)z+1} \tag{26}$$

If $|y_t'|<1$, we propose to use Fig. 5 because of smaller error to read y_t than x_t.

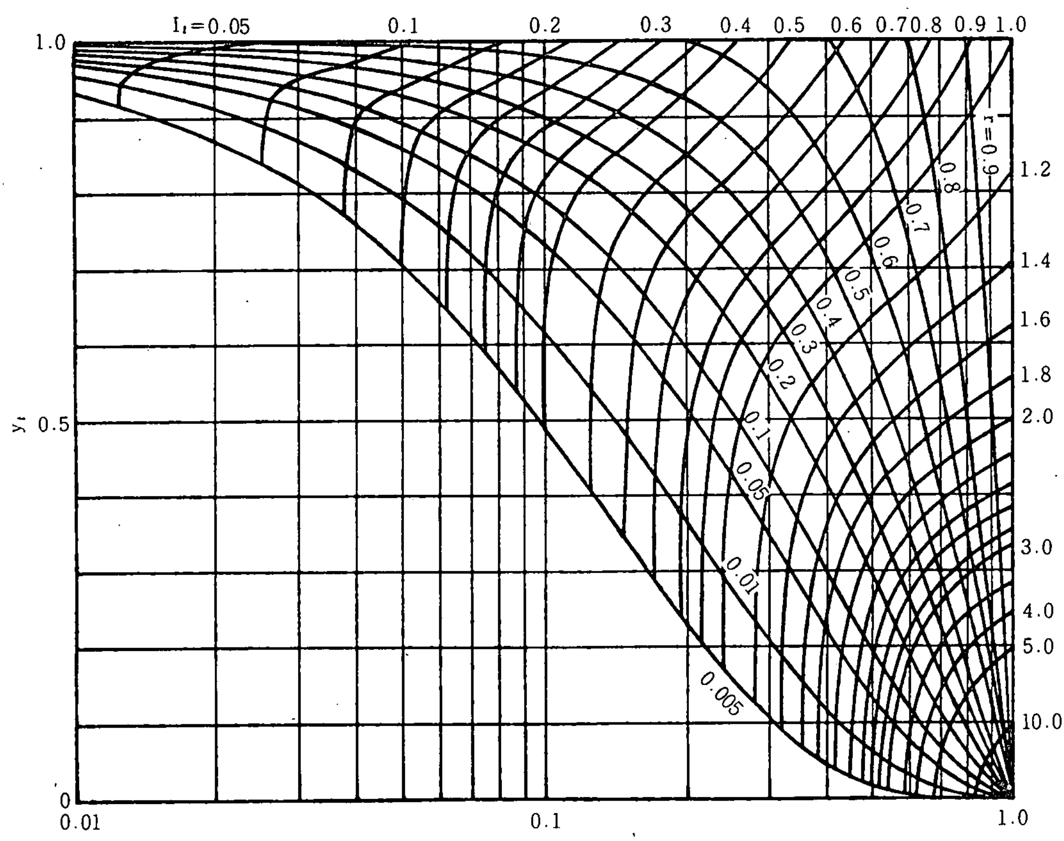

Fig. 5 Relation between y_t and z

Table－1 *The values of r and z by*

Yoshino river basin										Monobe river				
River basin No.	Neumerical method		Graphycal method		River basin No.	Neumerical method		Graphycal method		River basin No.	Neumerical method		Graphycal method	
	r_n	z_n	r_g	z_g		r_n	z_n	r_g	z_g		r_n	z_n	r_g	z_g
1	0.10	0.34	0.17	0.34	86	0.06	0.38	0.05	0.36	1	0.01	0.30	0.06	0.54
2	0.40	0.39	0.17	0.19	88	0.11	0.40	0.18	0.41	2	0.01	0.25	0.01	0.24
4	0.54	0.53	0.52	0.50	90	0.05	0.31	0.10	0.31	3	0.04	0.30	0.02	0.24
6	0.43	0.46	0.39	0.29	91	0.25	0.61	0.30	0.63	4	0.08	0.32	0.17	0.44
8	0.62	0.51	0.30	0.28	93	0.10	0.41	0.20	0.46	5	0.12	0.57	0.03	0.17
9	0.63	0.47	0.54	0.35	95	0.17	0.60	0.18	0.61	6	0.04	0.34	0.04	0.31
11	0.14	0.39	0.12	0.30	99	0.31	0.68	0.22	0.63	7	0.09	0.39	0.12	0.33
14	0.05	0.30	0.09	0.31	100	0.09	0.44	0.12	0.43	8	0.13	0.35	0.13	0.35
16	0.36	0.55	0.47	0.48	102	0.14	0.35	0.27	0.45	9	0.19	0.36	0.20	0.26
18	0.84	0.67			104	0.16	0.45	0.20	0.48	10	0.07	0.35	0.14	0.27
20	0.13	0.41	0.28	0.40	105	0.13	0.32	0.07	0.24	11	0.22	0.36	0.18	0.31
22	0.11	0.47			106	0.09	0.28	0.08	0.25	12	0.13	0.33	0.16	0.27
24	0.13	0.34	0.07	0.23	109	0.10	0.28	0.08	0.24	13	0.24	0.37	0.34	0.31
26	0.37	0.57	0.19	0.35	113	0.27	0.55	0.19	0.50	14	0.29	0.41	0.23	0.27
28	0.36	0.51	0.20	0.20	116	0.09	0.39	0.08	0.43	15	0.30	0.45	0.25	0.39
29	0.07	0.33	0.17	0.41	117	0.24	0.44	0.24	0.29	16	0.72	0.70	0.54	0.44
31	0.07	0.34	0.08	0.31	120	0.08	0.39	0.10	0.42	17	0.57	0.64	0.57	0.64
33	0.40	0.46	0.19	0.26	121	0.04	0.31	0.07	0.32	18	0.56	0.64	0.56	0.66
35	0.22	0.41	0.28	0.41	123	0.21	0.41	0.30	0.60	19	0.71	0.85		
37	0.19	0.45			127	0.13	0.32	0.12	0.25	20	0.20	0.53	0.15	0.48
44	0.13	0.37	0.10	0.31	131	0.06	0.29	0.15	0.37	21	0.37	0.47	0.28	0.38
46	0.08	0.35	0.13	0.32	132	0.71	0.64	0.45	0.37	22	0.27	0.39	0.17	0.25
48	0.21	0.37	0.08	0.25	136	0.21	0.43	0.10	0.35	23	0.09	0.31	0.14	0.26
50	0.14	0.37	0.11	0.25	138	0.24	0.45	0.18	0.32	24	0.12	0.30	0.04	0.16
51	0.05	0.32	0.13	0.34	140	0.16	0.40	0.15	0.42	25	0.54	0.46	0.53	0.45
53	0.07	0.36	0.16	0.41	142	0.23	0.50	0.34	0.49	26	0.16	0.42	0.16	0.30
54	0.15	0.64	0.09	0.61	144	0.13	0.45	0.15	0.45	27	0.27	0.42	0.47	0.33
55	0.12	0.67	0.05	0.56	145	0.08	0.34	0.11	0.28	28	0.61	0.57	0.50	0.35
57	0.41	0.71	0.22	0.54	146	0.12	0.37	0.16	0.27	29	0.13	0.34	0.15	0.26
59	0.02	0.31	0.06	0.48	148	0.78	0.70	0.10	0.14	30	0.38	0.60	0.34	0.58
60	0.08	0.50	0.12	0.41	150	0.57	0.62	0.60	0.49	31	0.25	0.43	0.63	0.47
61	0.20	0.54	0.28	0.60	152	0.32	0.50	0.43	0.49	32	0.16	0.39	0.26	0.30
65	0.74	0.82	0.78	0.88	154	1.34	0.73	0.05	0.10	33	0.02	0.25	0.06	0.17
67	0.08	0.38	0.08	0.36	156	0.50	0.40	0.23	0.26	34	0.17	0.39	0.18	0.40
69	0.10	0.29	0.15	0.28	158	0.69	0.47	0.22	0.19	35	0.15	0.37	0.13	0.25
71	0.22	0.44	0.23	0.44						36	0.26	0.46	0.29	0.34
73	0.21	0.39	0.18	0.38						37	0.29	0.42	0.20	0.21
75	0.14	0.34	0.20	0.26						38	0.19	0.46	0.40	0.41
76	0.50	0.55	0.38	0.38						39	0.10	0.41	0.20	0.62
78	0.15	0.44	0.12	0.34						40	0.09	0.32	0.08	0.27
79	0.18	0.42	0.32	0.45						41	0.06	0.26	0.18	0.31
81	0.27	0.55	0.37	0.55						42	0.04	0.28	0.08	0.33
83	0.12	0.39	0.18	0.40						43	1.66	1.10		

neumerical and graphcal method

basin					Watsuka river basin					Muroto peninsula				
River basin No.	Neumerical method		Graphycal method		River dasin No.	Neumerical method		Graphycal method		River basin NO.	Neumerical method		Graphycal method	
	r_n	z_n	r_g	z_g		r_n	z_n	r_g	z_g		r_n	z_n	r_g	z_g
44	0.63	0.61	0.66	0.59	1	0.10	0.26	0.03	0.19	1	0.21	0.57	0.26	0.51
45	0.20	0.35	0.18	0.26	2	0.04	0.19	0.01	0.09	2	0.04	0.41		
46	0.36	0.37	0.16	0.26	3	0.17	0.40	0.17	0.38	3	0.08	0.41	0.07	0.34
47	0.32	0.41	0.24	0.32	4	0.24	0.46	0.23	0.36	4	0.12	0.46	0.22	0.58
48	0.39	0.50	0.36	0.48	5	1.00	0.74			5	0.13	0.48	0.04	0.34
49	0.15	0.38	0.24	0.29	6	0.28	0.58	0.12	0.43	6	0.69	0.46	0.23	0.18
50	0.35	0.43	0.37	0.34	7	0.03	0.29	0.03	0.21	7	0.27	0.36	0.23	0.30
51	0.23	0.42	0.21	0.43	8	0.04	0.22	0.01	0.09	8	0.15	0.29	0.14	0.21
52	0.22	0.36	0.18	0.20	9	0.34	0.39	0.19	0.22	9	0.12	0.32	0.13	0.26
53	0.40	0.46	0.23	0.38	10	0.29	0.41	0.04	0.19	10	0.06	0.36	0.03	0.27
54	0.63	0.43	0.63	0.44	11	0.02	0.25	0.01	0.16	11	0.03	0.42	0.12	0.57
55	0.49	0.48			12	0.04	0.36	0.03	0.30	12	0.08	0.35	0.10	0.27
56	0.22	0.37	0.25	0.31	13	0.18	0.36	0.09	0.23	13	0.12	0.50	0.21	0.63
57	0.14	0.48	0.09	0.34	14	0.10	0.35	0.12	0.37	14	0.06	0.34	0.12	0.30
58	0.27	0.43	0.16	0.27	15	0.53	0.67	0.23	0.50	15	0.19	0.47	0.29	0.46
59	0.23	0.42	0.11	0.21	16	0.10	0.33	0.09	0.24	16	0.23	0.39	0.17	0.37
60	0.27	0.54	0.28	0.60	17	0.84	0.69	0.60	0.44	17	0.27	0.42	0.09	0.27
61	0.08	0.39	0.08	0.37	18	0.59	0.78			18	0.47	0.59		
62	0.19	0.47	0.20	0.47	19	0.09	0.51	0.04	0.39	19	0.06	0.32	0.17	0.28
63	0.10	0.38	0.07	0.25	20	1.07	0.98			20	0.01	0.23	0.16	0.37
64	0.44	0.59	0.20	0.46	21	0.27	0.48	0.42	0.50	21	0.49	0.35	0.42	0.31
65	0.19	0.47	0.19	0.47	22	0.22	0.58	0.23	0.52	22	0.53	0.47	0.31	0.34
66	0.11	0.42	0.18	0.45	23	0.35	0.44	0.15	0.18	23	0.13	0.30	0.11	0.23
67	0.16	0.36	0.19	0.37	24	0.12	0.43	0.32	0.55	24	0.22	0.36	0.17	0.34
68	0.16	0.35	0.16	0.35	25	1.55	1.43			25	0.20	0.32	0.29	0.34
69	0.11	0.29	0.16	0.22	26	0.08	0.44			26	0.24	0.38	0.18	0.31
70	0.08	0.33	0.08	0.33	27	0.32	1.35			27	0.13	0.38	0.14	0.28
71	0.30	0.44	0.22	0.34	28	1.17	0.65			28	0.27	0.37	0.52	0.38
72	0.35	0.48	0.29	0.48	29	0.19	0.81	0.11	0.61	29	0.37	0.35	0.17	0.21
73	0.89	0.72	0.69	0.49	30	0.41	0.84			30	0.57	0.51	0.57	0.39
74	0.71	0.57	0.71	0.57	31	0.03	0.34	0.06	0.39					
75	0.52	0.50	0.52	0.51	32	0.04	0.50	0.01	0.26					
76	0.55	0.51	0.55	0.55	33	0.09	0.44	0.01	0.26					
77	0.23	0.47	0.23	0.47	34	0.37	0.61	0.28	0.69					
78	0.22	0.55	0.30	0.72	35	0.25	0.60	0.04	0.41					
79	0.21	0.59	0.06	0.45	36	0.11	0.40	0.07	0.39					
80	0.27	0.62	0.18	0.61	37	0.04	0.39	0.11	0.51					
81	0.39	0.50	0.32	0.33	38	0.30	0.62	0.29	0.70					
82	0.24	0.47	0.35	0.41	39	0.39	0.85	0.13	0.56					
83	0.11	0.36	0.25	0.31	40	0.12	0.50	0.12	0.51					
84	0.64	0.74	0.60	0.80	41	0.08	0.38	0.07	0.42					
85	0.15	0.44	0.23	0.38	42	0.26	0.49	0.23	0.27					
					43	0.37	0.39	0.22	0.25					

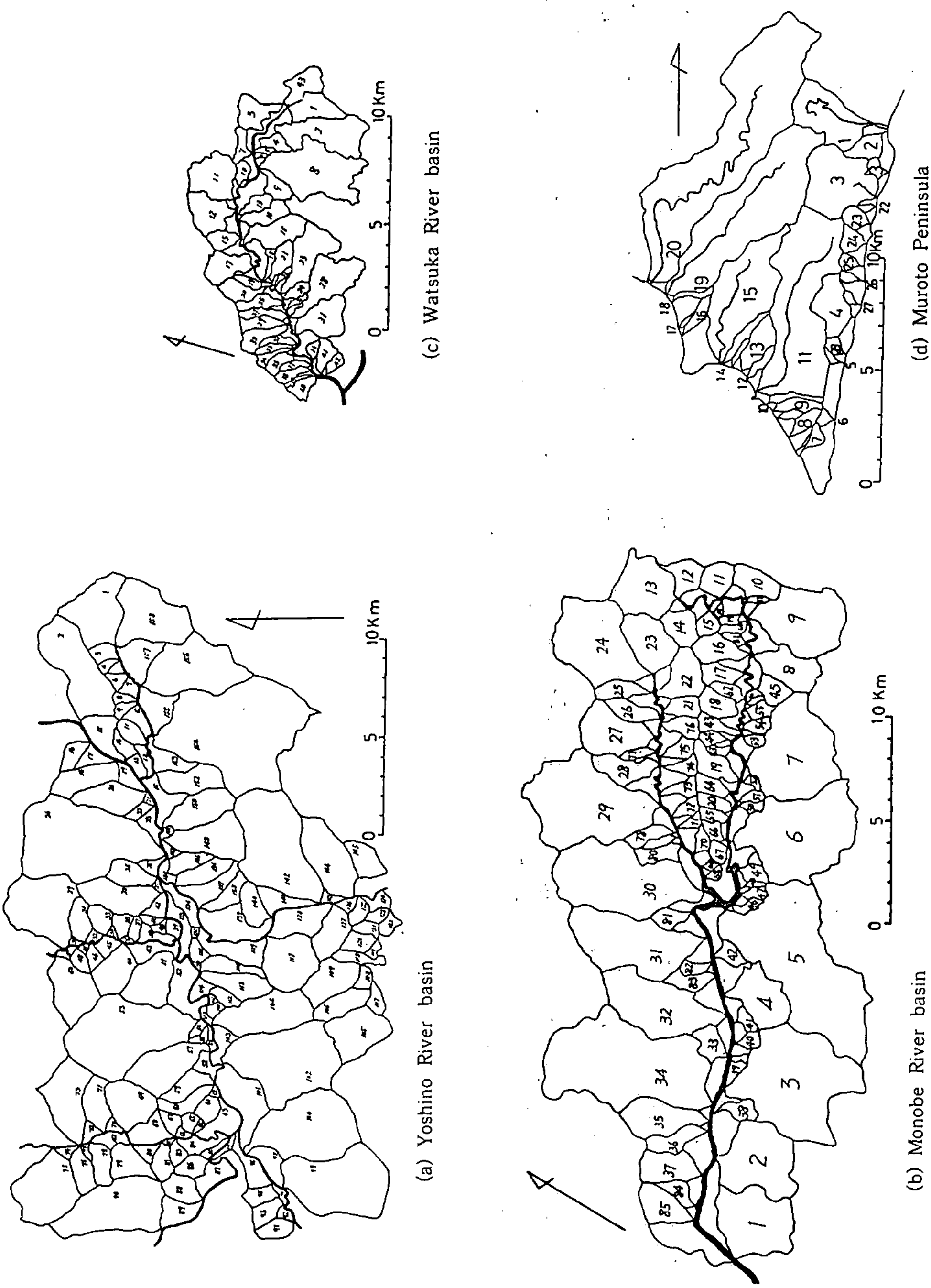

Fig. 6 Location of analysis area.

Values of constants on several example

We obtained the constants r and z of several chosen basin Yoshino (Kochi), Monobe (Kochi), Watsuka (Kyoto) river basin and Muroto peninsula (Kochi) in which exisited 2~5 order branch stream basins. There were 236 basins totally, but we could not determine the inflection point in 16 basins.

Locations of basin and r_g, z_g or r_n, z_n of each basins, where, sufix g or n means to be obtained by graphycal method or meunerical method are shown in Fig. 6 (a)~(d) and table 1.

On the computation by neumerical method, we used FACOM 270-20/30 belong Kochi University. Two examples of small and large defference by both method are shown in Fig. 7 (a)~(c).

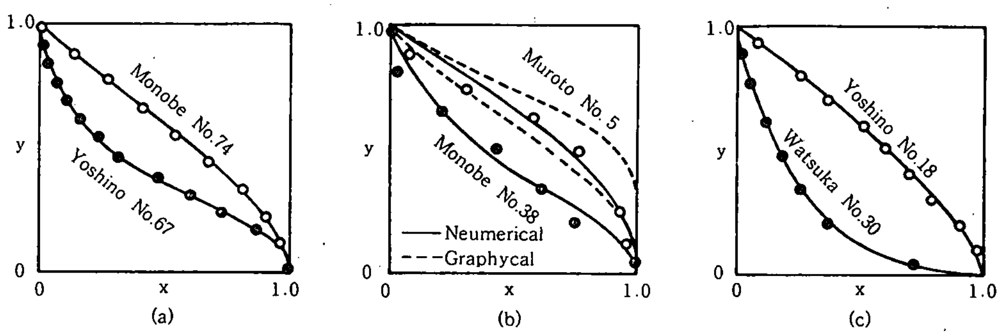

Fig. 7 Some example of percentage hypsometric curve by neumerical and graphycal method.
(a) Nearly equal z_n and z_g, r_n and r_g.
(b) Large difference between neumerical and graphycal method.
(c) No inflection point in $0<x<1.0$.

Consideration and results

Relation between r_g, z_g and r_n, z_n and expressed the proportional constants A or B as follows,

$$r_g = A \cdot r_n, \quad z_g = B \cdot z_n$$

A or B was obtained by least square method through the original point (0, 0) of r and z axis in Fig. 8 (a)~(d).

As a result of total basins, we obtained A=0.84, B=0.86 as atemparary standerd. We may conclude at this point that the values of constants obtained by graphycal method are about 85% of that obtaind by neumerical method.

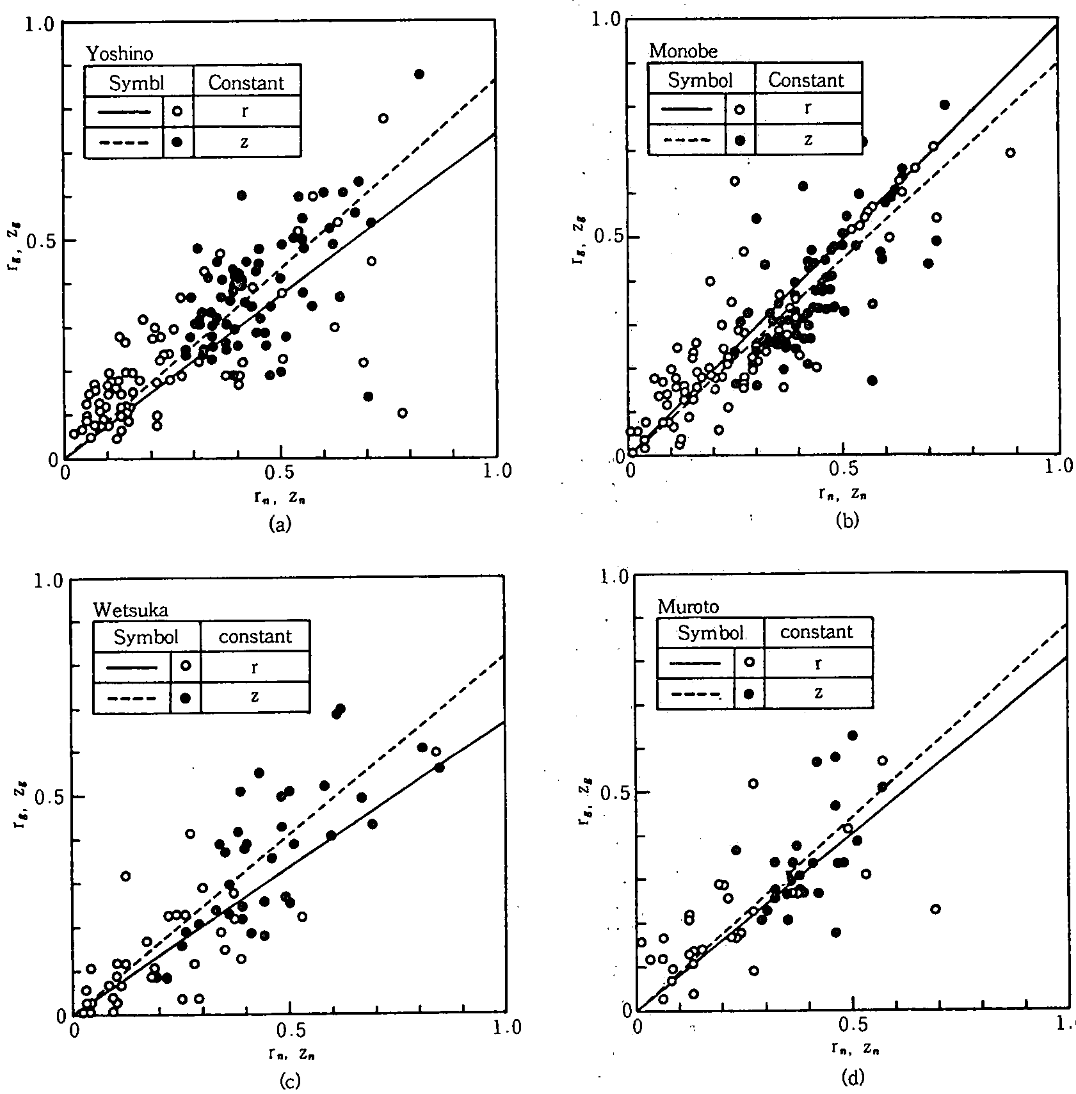

Fig. 8 Relation between r_g, z_g and r_n, z_n.

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