Generalization of a Limit Theorem of Mercer

BY Torao Uchida.

(Seminal of Mathematics, Liberal Arts Faculty, Kochi University)

Introduction and Theorems

Cauchy has proved that if

$$x_{n+1} - x_n \to s \qquad (1)$$

as $n \to \infty$, then also

$$\frac{x_n}{n} \to s \qquad (2)$$

or, what amounts to the same thing, that if

$$s_n \to s,$$

then

$$\frac{s_1 + s_2 + \cdots s_n}{n} \to s.$$

Of course, the converse of these theorems are in general untrue.

From (1) and (2) it follows that

$$x_{n+1} - x_n - a\left(\frac{x_n}{n}\right) \to (1-a)\, s \qquad (3)$$

It has been shown by Mercer that, if a is real and less than 1, (1) and (2) are both consequences of (3); but if a is 1, this is not the case. Afterwards G. H. Hardy has proved the following theorems.

Theorem 1. If a is a reel or complex constant whose real part is not equal to unity, and

$$f'(x) - \frac{a}{x} f(x) = o(1), \qquad (4)$$

then $\quad f(x) = c\, x^a + o(x),$

where c is a constant. If the real part of a is less than unity, c is zero.

Theorem 2. If

$$f(n+1) - f(n) - \frac{a}{n} f(n) = o(1),$$

and $\alpha \neq 1$, *then*

$$f(n) = c\, \frac{\Gamma(n+a)}{\Gamma(n)} + o(n),$$

where c is a constant. If $\alpha < 1$, *c is zero*

At first taking the more general ones

$$f'(x)-\frac{q(x)}{x+a}f(x)=o(1),$$

instead of (4), we will prove the theorems corresponding to Theorem 1.

Theorem 3.1. *If* $\overline{\lim}\, q(x)<1$

and

$$f'(x)-\frac{q(x)}{x+a}f(x)=o(1) \qquad (5)$$

then

$$f(x)=o(x).$$

Theorem 3.2 *If* $\underline{\lim}\, q(x)>1$

and

$$f'(x)-\frac{q(x)}{x+a}f(x)=o(1)$$

then

$$f(x)=l\exp\left(\int^{x}\frac{q(x)}{x+a}dx\right)+o(x).$$

Theorem 4.1. *If* $\overline{\lim}\, q(x)<1$

and

$$f'(x)-\frac{q(x)}{x+a}f(x)=m+o(1), \qquad (6)$$

then

$$f(x)=me^{Q(x)}\int^{x}e^{-Q(x)}dx+o(x),$$

where $Q(x)=\int^{x}\frac{q(x)}{x+a}dx.$

Theorem 4.2. *If* $\underline{\lim}\, q(x)>1$

and

$$f(x)-\frac{q(x)}{x+a}f(x)=m+o(1)$$

then

$$f(x)=le^{Q(x)}+me^{Q(x)}\int^{x}e^{-Q(x)}dx+o(x).$$

Proof of Theorem 3.1.

In order to prove Theorem 3.1. we put

$$f(x) = \phi(x)\, e^{Q(x)}$$

where $Q(x) = \int^{x} \frac{q(x)}{x+a}\, dx,$

and substitute in (5), then we have

$$\phi'(x)\, e^{Q(x)} + Q'(x).\, \phi(x)\, e^{Q(x)} - \frac{q(x)}{x+a}\, \phi(x)\, e^{Q(x)} = o(1),$$

$$\phi'(x)\, e^{Q(x)} = o(1).$$

therefore,

$$\phi'(x) = o(e^{-Q(x)}).$$

When we put $\overline{\lim}\, q(x) = b' < 1$, we can take b such that $q(x) \leqq b$, where $b' < b < 1$.

Now we suppose $x+a > o$ and $q(x) \leqq b$ for sufficiently large value of x and then we have

$$\frac{q(x)}{x+a} \leqq \frac{b}{x+a}$$

Therefore

$$Q(x) = \int^{x} \frac{q(x)}{x+a}\, dx < \int^{x} \frac{b}{x+a}\, dx,$$

and we have

$$\int^{\infty} e^{-Q(x)}\, dx > \int^{x} \frac{dx}{exp\left(\int^{x} \frac{b}{x+a}\, dx\right)} = \int^{\infty} \frac{dx}{A(x+a)^{b}} = \infty.$$

Consequently

$$\phi(x) = c + o\left(\int^{x} \frac{dx}{exp\left(\int^{x} \frac{b}{x+a}\, dx\right)}\right) = o\left(\int^{x} \frac{dx}{exp\left(\int^{x} \frac{b}{x+a}\, dx\right)}\right),$$

$$f(x) = o\left(\int^{x} e^{-Q(x)}\, dx\right).\, e^{Q(x)} = o\left(e^{Q(x)} \int^{x} e^{-Q(x)}\, dx\right).$$

Now, there is a constant A such that

$$e^{-Q(x)} < A\left(1 - \frac{x}{x+a}\, q(x)\right) e^{-Q(x)} \tag{7}$$

Therefore

$$\int^{x} e^{-Q(x)}\, dx < A\, x\, e^{-Q(x)} + B, \tag{8}$$

and

$$e^{Q(x)} < \exp\left(\int^{x} \frac{b}{x+a}\,dx\right) = c\,(x+a)^{b} \tag{9}$$

(7), (8) and (9) give us $f(x) = o(x)$.

Proof of Theorem 3.2.

When we put $\lim q(x) = b' < 1$, we can take b such that $q(x) \geqq b$, where $b' > b > 1$. Then we have

$$\int^{\infty} e^{-Q(x)} dx \leqq \int^{\infty} \exp\left(-\int^{x} \frac{b}{x+a} dx\right) = \int^{\infty} \frac{A}{(x+a)^{b}}\, dx.$$

Therefore $\int^{\infty} \phi'(x)\,dx$ converges to a Value l, so that

$$\phi(x) = l - \int_{x}^{\infty} o\left(e^{-Q(x)}\right) dx = l - o\left(\int_{x}^{\infty} e^{-Q(x)} dx\right)$$

and then

$$f(x) = l e^{Q(x)} - o\left(e^{Q(x)} \int_{x}^{\infty} e^{-Q(x)} dx\right). \tag{10}$$

Now there is a constant A such that

$$e^{-Q(x)} < A\left(\frac{x}{x+a} q(x) - 1\right) e^{-Q(x)},$$

therefore

$$\int_{x}^{\infty} e^{-Q(x)} dx < A\left(\int_{x}^{\infty} \frac{x}{x+a} q(x) e^{-Q(x)} dx - 1\right) e^{-Q(x)} dx = A e^{-Q(x)} x, \tag{11}$$

that is

$$\int_{x}^{\infty} e^{-Q(x)} dx < A\, e^{-Q(x)} x. \tag{12}$$

Combining (10), (11) and (12), we get

$$f(x) = l \exp\left(\int^{x} \frac{q(x)}{x+a} dx\right) + o(x).$$

proof of Theorem 4.1 and 4.2.

Let us put

$$\mathrm{P}(x) = f(x) - m\, e^{Q(x)} \int^{x} e^{-Q(x)} dx,$$

and substitute in (6), then we get

$$p'(x) - \frac{q(x)}{x-a} \, p(x) = o(1),$$

therefore we can reduce these theorems into Theorems 3.1 and 3.2.

Remark 1. Theorems 3 and 4 may be generalized by replacing $O(1)$ in the expression from which we start, by $o(\varphi)$, where φ is an increasing function of x.

Remark 2. We can extend the above results to the differential equation of higher order. For example, we have the following theorem.

Theorem 5. *if the roots $r(x)$, $s(x)$ of the equaton*

$$y(y-1) - p(x)y + q(x) = o$$

satisfy the conditions

$$\underline{Lim}\ r(x) > 2, \qquad \underline{Lim}\ s(x) > 2$$

and

$$f''(x) - \frac{p(x)}{x-a} f'(x) - \frac{q(x)}{(x-a)^2} f(x) = o(1),$$

then

$$f(x) = l \exp\left(\int^x \frac{r(x)}{x-a}\, dx\right) + m \exp\left(\int^x \frac{s(x)}{x-a}\, dx\right) - o(x^2)$$

where l and m are constants.

Remark 3. We can generalize the Theorem 2 in the similar method of Theorem 3.

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