CERTAIN SELECTION PROCEDURES BASED ON ORDERED STATISTICS

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1. Summary.

Several statistical procedures are presented for selecting a subset from k given exponential distributions which contain t-best $(k \geqq t)$ populations. These procedures are constructed by the statistics which are based on censored data from respective exponential distributions. In the cases when the observations become available in ordered manner, usual statistical problems do not make use of the original random samples. The practical applications which need the censored data are represented by Esptein and Sobel [6] and etc.

Now, the ranking or selection problems which are prsented by Bechhofer [1] have been studied are developed by many authors. The author [7] in this paper presented one general selection procedure for the best population among the family of one parameter exponential distributions. Another selection methods by means of the nonparametric methods have been studied by the authors such as Lehmann [6], Rizvi and Sobel [9] and Sobel [14] and etc.

On one hand, no statistical selection probelems which make use of the procedures based on censored data were studied before. At these points of view, our main object in this paper is to challenge an attention to these statistical selection problems by means of censored data. However a number of kinds of statistics based on censored data for the exponential distributions are studied by the authors such as Esptein and Sobel [2] and [4], Esptein [3], Kulldorff [5], Ogawa [8], Saleh [10], Sarhan [11], Sarhan, Greenberg and Ogawa [12], Sarhan and Greenberg [13], [15], and etc.

Thus we discuss in Section 1 one selection problem whose procedure depends on the best linear unbiased estimates by Sarhan and Greenberg [13] for the scale parameters in the case when a set of one parameter exponential distributions are presented, and discuss the problem whose procedure depends on the unbiased estimate by Esptein and Sobel [4], making use of Chi-square distributions.

In Section 2, and 3 we discuss the selection problems whose procedures depend on the best linear estimates by Sarhan and Greenberg [13] for the scale and location parameters in the case when two parameter exponential distributions are presented.

In Section 4, we present a selection procedure based on ordered statistics whose respective dentisty function is uniformly.

2. One parameter exponential distributions.

2.1. The scale parameter case when $x_{(r_1+1)}, \cdots, x_{(n-r_2)}$ are used.

Let

(2.1.1.) $$f_\sigma(x) = \sigma^{-1} \exp(-x/\sigma), \quad 0 \leqq x < \infty, \quad 0 < \sigma,$$

and let $y_1 \leqq y_2 \leqq \cdots \leqq y_n$ be the ordered statistics of $x_1, x_2, \cdots, x_n$ whose respective density is given by (2.1.1.). By use of $y_{r_1} \leqq y_{r_1+1} \leqq \cdots \leqq y_{n-r_2}$, let us put

(2.1.2) $$\sigma^* = \{Dy_{r_1+1} + \sum_{i=r_1+2}^{n-r_2-1} y_i + (r_2+1)y_{n-r_2}\}/K,$$

where we put

(2.1.3) $$K = K_2^2/D_2 + (n-r_1-r_2-1)$$

(2.1.4) $$D = K_1/D_1 - (n-r_1-1)$$

and where

(2.1.5) $$K_i = n^{-1} + (n-1)^{-1} + \cdots + (n-r_i)^{-1}, \ (i = 1, 2)$$

(2.1.6) $$D_i = n^{-2} + (n-1)^{-2} + \cdots + (n-r_i)^{-2}, \ (i = 1, 2).$$

Then the statistic σ^* is the best linear unbiased estimate of σ, by [13].

In order to make the density function of σ^*, we give the joint density function of $y_{r_1}, y_{r_1+1}, \cdots, y_{n-r_2}$:

(2.1.7) $$\xi(y_{r_1}, y_{r_1+1}, \cdots, y_{n-r_2}) = \frac{n!}{r_1!\, r_2!}[F_\sigma(y_{r_1+1})]^{r_1}[1-F_\sigma(y_{n-r_2})]^{r_2} \times \prod_{i=r_1+1}^{n-r_2} f_\sigma(y_i),$$

where

(2.1.8) $$F_\sigma(y) = \int_0^y f_\sigma(t)dt, \qquad (y \geqq 0).$$

By mean of the transformation

(2.1.9) $$y/\sigma = s, \quad y_i/\sigma = t_i, \quad (i = r_1+1, r_1+2, \cdots, n-r_2)$$

the equation (2.1.7) reduces to

(2.1.10) $$\eta(t_{r_1+1}, t_{r_1+2}, \cdots, t_{n-r_2}) = n!(r_1!r_2!)^{-1}[H(t_{r_1+1})]^{r_1} \times [1-H(t_{n-r_2})]^{r_2} \prod_{i=r_1+1}^{n-r_2} h(t_i),$$

where

(2.1.11) $$H(x) = \int_0^x h(t)dt \ \text{ and } \ h(t) = \exp(-t), \quad (t \geqq 0).$$

Let Π_i be the population with density function $f\sigma_i(x)$, $(x \geqq 0)$, which was defined by (2.1.11). Let us put

(2.1.12) $$\Omega(t) = \{\sigma = (\sigma_0, \sigma_1, \cdots, \sigma_k) \,|\, \sigma_i = r_i\sigma_0, \quad \sigma_j = \delta_j\sigma_0, \quad i = 1, 2, \cdots, t; \\ j = t+1, t+2, \cdots, k\},$$

where for the preassigned positive constants $\delta^* > 1 > r^*$, we assume that

(2.1.13) $$r_i \leqq r^* < 1 < \delta^* \leqq \delta_j, \quad i = 1, 2, \cdots, t; j = t+1, t+2, \cdots, k.$$

Procedure R_1: If $\sigma_0^* \geqq \sigma_i^* (\sigma_j^* \geqq \sigma_0^*)$, then we decide that $\Pi_i \in S (\Pi_j \in I)$, respectively.

Let (CD) be such stochastic event that

$$\Pi_i \in S, \quad \Pi_j \in I, \ i = 1, 2, \cdots, t; j = t+1, t+2, \cdots, k,$$

by use of Procedure R_1, where I and S mean the set of populations whose parameters are of type $r_i\sigma_0$ and $\delta_j\sigma_0$, respectively. Then we evaluate the probability $P\{(CD)\,|\,\Omega(t), R_1\}$ by which the correct selection is done by use of the procedure R_1, for each $\boldsymbol{\sigma}\in\Omega(t)$. By mean of the transformation (2.1.9) of variables we have

(2.1.14) $$\sigma^*/\sigma = (Dt_{r_1+1}+t_{r_1+2}+\cdots+t_{n-r_2-1}+(r_2+1)t_{n-r_2})/K\,,$$

therefore from (1.1.14)

(2.1.15) $$\begin{aligned} t_{r_1+1} &= (K\sigma^*/\sigma-t_{r_1-2}-\cdots-t_{n-r_2-1}-(r_2+1)t_{n-r_2})/D \\ t_i &= t_i \qquad (i=r_1+2,\ \cdots,\ n-k)\,. \end{aligned}$$

The Jacobian of transformation is given by

(2.1.16) $$J=K/(\sigma D)\,,$$

where the constants K and D were given by (2.1.3) and (2.1.4), respectively. Hence we have

(2.1.17) $$\begin{aligned} &\varphi(\sigma^*,\ t_{r_1+2},\ \cdots,\ t_{n-r_2}) \\ &=\frac{n!\,K}{r_1!\,r_2!\,\sigma D}\left[H\left(\frac{K\sigma^*}{D\sigma}-\frac{1}{D}(t_{r_1+2}+\cdots+t_{n-r_2-1})-\frac{r_2+1}{D}t_{n-r_2}\right)^{r_1}[1-H(t_{n-r_2})]^{r_2}\right. \\ &\quad\left.\times\exp\left(-\frac{K\sigma^*}{D\sigma}-\frac{D-1}{D}t_{r_1+2}-\cdots-\frac{D-1}{D}t_{n-r_2-1}-\frac{D-r_2-1}{D}t_{n-r_2}\right)\right], \end{aligned}$$

Therefore we have

(2.1.18) $$g(\sigma^*)=\int_0^\infty\cdots\int_0^\infty g(\sigma^*,\ t_{r_1+2},\ \cdots,\ t_{n-r_2})\prod_{i=r_1+1}^{n-r_2}dt_i\,.$$

This is the density function of $\sigma^{*'}$ defined by (2.1.2).

Now let us obtain the joint density function of $k+1$ estimates defined by (2.1.2) for k experimental populations and one control populations. Since in this case $k+1$ estimates $\sigma_0^*,\ \sigma_1^*,\ \cdots,\ \sigma_k^*$ are mutually independent we have

(2.1.19) $$P\{(CD)\,|\,\Omega(t),R_1\}=\underset{\substack{\{\sigma_0^*\geqq\sigma_i^*.\ i=1,2,\cdots,t;\\ \sigma_j^*\geqq\sigma_,^*,j=t+1,\cdots,k\}}}{\int\cdots\int}\prod_{i=0}^{k}[g(\sigma_i^*)d\sigma_i^*]\,.$$

By use of the transformation

(2.1.20) $$\sigma_i^*/\sigma_i=\hat\sigma_i,\quad i=0,1,\cdots,k\,,$$

the equation (2.1.19) redices to

(2.1.21) $$\int_0^\infty\prod_{i=1}^{t}G_i(\hat\sigma_0/\theta_i)\prod_{j=t+1}^{k}[1-G_j(\hat\sigma_0/\theta_j)]g(\hat\sigma_0)d\hat\sigma_0\,,$$

where $\theta_p=\sigma_p/\sigma_0$, $p=1,2,\cdots,k$ and

(2.1.22) $$\begin{aligned} G_i(x)=\int_0^x\Bigg\{\int_0^\infty\cdots\int_0^\infty\frac{n!\,K}{r_1!\,r_2!D}&\left[H\left(\frac{K}{D}\hat\sigma_i-\frac{t_{r_1+2}}{D}-\frac{t_{n-r_2-1}}{D}-\frac{r_2+1}{D}t_{n-r_2}\right)\right]^{r_1} \\ \left[1-H(t_{n-r_2})\right]^{r_2}&\exp\left(-\frac{K}{D}\hat\sigma_i-\frac{D-1}{D}t_{r_1+1}-\cdots-\frac{D-1}{D}t_{n-r_2-1}-\frac{D-r_2-1}{D}t_{n-r_2}\right] \\ \times\prod_{j=r_1+2}^{n-r_2}dt\Bigg\}d\hat\sigma_i,\ (i=1,2,\cdots,k)\,.& \end{aligned}$$

Let us define that

(2.1.23) $\Omega^*(t) = \{\boldsymbol{\sigma} \mid \sigma_i = r^*\sigma_0,\ \sigma_j = \delta^*\sigma_0,\ i = 1, 2, \cdots, t; j = t+1, \cdots, k\}$,

where it was that

(2.1.24) $$r_i \leqq r^* < 1 < \delta^* \leqq \delta_j,\ i = 1, 2, \cdots, t; j = t+1, \cdots, k .$$

Now for each i in $t \geqq i \geqq 1$ we have

(2.1.25) $$\begin{aligned} G(\hat{\sigma}_0/\sigma_i) &= G_i(\hat{\sigma}_0/r_i) \\ &= \int_0^{\hat{\sigma}_0/r_i}\left\{\int_0^\infty \cdots \int_0^\infty \frac{n!\,K}{r_1!\,r_2!\,D}\left[H\left(\frac{K}{D} - \frac{t_{r_1+2}}{D} - \cdots - \frac{t_{n-r_2-1}}{D} - \frac{r_2+1}{D}t_{n-r_2}\right)\right]^{r_1}\right. \\ &\quad \times \left[1 - H(t_{n-r_2})\right]^{r_2} \exp\left(-\frac{K}{D}\hat{\sigma}_i - \frac{D-1}{D}t_{r_1+2} - \cdots - \frac{D-1}{D}t_{n-r_2-1} - \frac{D-r_2-1}{D}t_{n-r_2}\right. \\ &\quad \left. \times \prod_{i=r_1+2}^{n-r_2} dt\right\} d\hat{\sigma}_0 \\ &\geqq G_i(\hat{\sigma}_0/r^*) , \end{aligned}$$

and similary for each j in $t+1 \leqq j \leqq k$ we have

(2.1.25) $$1 - G_j(\hat{\sigma}_0/\theta_j) \geqq 1 - G_j(\hat{\sigma}_0/\delta^*) .$$

Hence we obtain the following theorem:

Theorem 2.1. *Under the notations and conditions stated above, we have*

(2.1.27) $$\begin{aligned} \underset{\substack{\{r_i \leqq r^* < 1 < \delta^* \leqq \delta_j ,\\ i=1,2,\cdots,t; j=t+1,\cdots,k\}}}{Inf} P\{(CD) \mid \Omega(t), R_1\} &= P\{(CD) \mid \Omega^*(t), R_1\} \\ &= \int_0^\infty \left[G(\hat{\sigma}_0/r^*)\right]^{r_1} \left[1 - G(\hat{\sigma}_0/\delta^*)\right]^{k-r_1} g(\hat{\sigma}_0) d\hat{\sigma}_0 , \end{aligned}$$

where $G(y)$ was defined by (2.1.25).

2.2. The scale parameter cases when $x_{(1)}, x_{(2)}, \cdots, x_{(m)}$ are used.

Let Π_i be the exponential population whose density function is given by

(2.2.1) $$f\sigma_i(x) = \exp(-x/\sigma_i)/\sigma_i,\quad 0 \leqq x < \infty,\quad 0 < \sigma_i ,$$

$i=0, 1, \cdots, k$, respectively. Let the transformation be

(2.2.2) $$((2m-2)\hat{\sigma}_i/\sigma_i)/((2m-2)\hat{\sigma}_0/\sigma_0) = F_i = \hat{\sigma}_i/(\hat{\sigma}_0\theta_i) ,$$

where the statistic $(2m-2)\hat{\sigma}_i/\sigma_i$ distributes Chi-square distribution with $2m-2$ degrees of freedom. Let $x_{(1)} < x_{(2)} < \cdots < x_{(m)}$ be the first m $(n>m)$ ordered statistic of $x_1, x_2, \cdots, x_n$ which have the respective density function defined by (2.2.1). Let us put that

(2.2.3) $$\hat{\sigma}_i = \sum_{i=1}^{m-1}(n-i)(x_{(i+1)} - x_{(i)})/(m-1) ,$$

$i=0, 1, \cdots, k$. Then it was proved by Esptein and Sobel [4] that the statistic $\hat{\sigma}_i$ is the minimum variance unbiased estimate of σ_i, $i=1, 2, \cdots, k$, respectively. Now let us give the joint density function of $(F_1, F_2, \cdots, F_k)$ by use of that of $(\hat{\sigma}_0, \hat{\sigma}_1, \cdots, \hat{\sigma}_k)$. Since $(2m-2)$

$\hat{\sigma}_i/\sigma_i$ $i=0, 1, \cdots, k$ are mutually independent and each statistic has Chi-square distribution with $2m-2$ degrees of freedom, we have

(2.2.4)
$$g(\hat{\sigma}_0, \hat{\sigma}_1, \cdots, \hat{\sigma}_k)\prod_{i=0}^{k} d\hat{\sigma}_i = \prod_{i=0}^{k}\left[\frac{1}{\Gamma(m-1)}\left(\frac{(m-1)\hat{\sigma}_i}{\sigma_i}\right)^{m-2}\exp\left(-\frac{(m-1)\hat{\sigma}_i}{\sigma_i}\right)d\left(\frac{(m-1)\hat{\sigma}_i}{\sigma_i}\right)\right].$$

The following tranformation

(2.2.5)
$$F_i\theta_i = \hat{\sigma}_i/\sigma_i, \quad i = 1, 2, \cdots, k$$

has the next Jacobian:

(2.2.6)
$$J = |\partial\hat{\sigma}_i/\partial F_j| = (\hat{\sigma}_0)^k,$$

where $\theta_i=\sigma_i/\sigma_0$, $i=1, 2, \cdots, k$, therefore we have the next joint density function of $F_1, F_2, \cdots, F_k$:

(2.2.7)
$$h(F_1, F_2, \cdots, F_k) = \frac{(m-1)^{k+1}}{[\Gamma(m-1)]^{k+1}}\prod_{i=0}^{k}\left(\frac{F_i}{\sigma_0}\right)^{m-1}\int_0^\infty(\hat{\sigma}_0)^{(k+1)(m-1)} \times \exp\left(-\frac{\hat{\sigma}_0}{\sigma_0}(1+F_1+\cdots+F_k)\right)d\hat{\sigma}_0 .$$

Again the next transformation

(2.2.8)
$$(m-1)(1+F_1+\cdots+F_k)\hat{\sigma}_0/\sigma_0 = s$$

makes the equation (2.2.7) to

(2.2.9)
$$h(F_1, F_2, \cdots, F_k) = \frac{\sigma_0}{m-1}\frac{\Gamma(km+m-k)}{[\Gamma(m-1)]^{k+1}}\frac{\prod_{i=1}^{k} F_i^{m-1}}{(1+F_1+\cdots+F_k)^{km+m-k}}.$$

Procedure R_2: *If $\hat{\sigma}_0 \geqq \hat{\sigma}_i$ ($\hat{\sigma}_j > \hat{\sigma}_0$), then we decide that $\Pi_i \in S$ ($\Pi_j \in I$), respectively.*
This procedure means that if $F_i \leqq r^{-1}(F_j > \delta_j^{-1})$, then $\Pi_i \in S(\Pi_i \in I)$, respectively. Hence we obtain

(2.2.10)
$$P\{(CD)\,|\,\Omega(t), R_2\} = \underset{\substack{\{F_i \leqq r_i^{-1}, i=1, 2, \cdots, t;\\ F_j > \delta_j^{-1}, j=t+1, \cdots, k\}}}{\int\cdots\int} h(F_1, F_2, \cdots, F_k)\prod_{i=1}^{k} dF_i .$$

From (2.2.10) we have the following theorem:

Theorem 2.2.1 *Under the notions and conditions stated above we have*

(2.2.11)
$$\underset{\substack{\{r_i \leqq r^* < 1 < \delta^* \leqq \delta_j,\\ i=1, 2, \cdots, t; j=t+1, \cdots, k\}}}{Inf} P\{(CD)\,|\,\Omega(t), R_2\} = P\{(CD)\,|\,\Omega^*(t), R_2\} .$$

The proof of this theorem follows by means of the similar notions in (2.1.24) and (2.1.25).

Nextly, we give the another representation of (2.2.9) which is the joint density functhon of $F_1, F_2, \cdots, F_k$ as follows:

(2.2.12) $$P\{(CD)\mid \Omega(t), R_2\} = \int_0^\infty \prod_{i=1}^{t} K(m-1, s_0/r_i) \prod_{j=t+1}^{k} [1-K(m-1, s_0/\delta_j)]dK(m-1, s_0),$$

where

(2.2.13) $$K(m-1, s_0/r_i) = \frac{1}{\Gamma(m-1)}\int_0^{s_0/r_i} t^{m-2}e^{-t}dt, \ i = 1, 2, \cdots, t.$$

Hence the result of theorem 2.2. follows similary.

In what follows, we give the saymptotic representation as m becomes large. Since a set of statistics

(2.2.14) $$T_i = (\hat{\sigma}_i-\sigma_i)/(2\sigma_i), \quad i = 0, 1, \cdots, k$$

are mutually independent and the respective density function has $N(0, 1)$, we obtain the following asymptotic representation:

(2.2.15) $$\begin{aligned} &P\{(CD)\mid \Omega(t), R_2\} \\ &\sim \int_{-\infty}^{\infty} \prod_{i=1}^{t} \Phi\left(\frac{1}{2}\left[\left\{\frac{\hat{\sigma}_0-\sigma_0}{\sigma_0}+1\right\}/r_i-1\right]\right) \\ &\prod_{j=t+1}^{k}\left[1-\Phi\left(\frac{1}{2}\left[\left\{\frac{\hat{\sigma}_0-\sigma_0}{\sigma_0}+1\right\}/\delta_j-1\right)\right]\right)\phi\left(\frac{\hat{\sigma}_0-\phi_0}{2\sigma_0}\right)d\left(\frac{\hat{\sigma}_0-\sigma_0}{2\sigma_0}\right). \end{aligned}$$

By means of the transformation

(2.2.16) $$(\hat{\sigma}_0/\sigma_0-1)/2 = t,$$

the right hand side of (2.2.15) reduces to

(2.2.17) $$\begin{aligned} &\int_{-\infty}^{\infty} \prod_{i=1}^{t} \Phi\left(\frac{1}{2}\left[\frac{2t+1}{r_i}-1\right]\right)\prod_{j=t+1}^{k}\left[1-\Phi\left(\frac{1}{2}\left[\frac{2t+1}{\delta_j}-1\right)\right]\phi(t)dt \\ &\geqq \int_{-\infty}^{\infty}\left[\Phi\left(\frac{t}{r^*}+\frac{1}{2}\left(\frac{1}{r^*}-1\right)\right)\right]^t\left[1-\Phi\left(\frac{t}{\delta^*}+\frac{1}{2}\left(\frac{1}{\delta^*}-1\right)\right)\right]^{k-t}\phi(t)dt \\ &= \int_{-\infty}^{\infty}\left[\Phi\left(\tau t+\frac{\tau-1}{2}\right)\right]^t\left[1-\Phi\left(\Delta t+\frac{\Delta-1}{2}\right)\right]^{k-t}\phi(t)dt, \end{aligned}$$

where

(2.2.18) $$1/r^* = \tau > 1 \ \ and \ \ 1/\delta^* = \Delta < 1.$$

Therefore we have the following:

Theorem 2.2.2. *As the common sample size m increases infinitely,* we obtain

(2.2.19) $$\begin{aligned} &P\{(CD)\mid \Omega(t), R_2\} \\ &\sim \int_{-\infty}^{\infty} \prod_{i=1}^{t} \Phi\left(\frac{1}{2}\left(\frac{2t+1}{r_i}-\right)\right)\prod_{j=t+1}^{k}\left[1-\Phi\left(\frac{1}{2}\left(\frac{2t+1}{\delta_i}-1\right)\right)\right]^{k-t}\phi(t)dt \\ &\geqq \int_{-\infty}^{\infty}\left[\Phi(\tau t+(\tau-1)/2)\right]^t\left[1-\Phi\left(\Delta t+\frac{\Delta-1}{2}\right)\right]^{k-t}\phi(t)dt, \end{aligned}$$

where

(2.2.20) $$\tau = 1/r^* > 1 \ \ and \ \ \Delta = 1/\delta^* < 1.$$

3. Two parameter eponential distributions.

3.1. The scale parameter case when $x_{(r_1+1)}, \cdots, x_{(n-r_2)}$ are used.

Let $x_{(1)} \leqq \cdots \leqq x_{(n)}$ be the ordered statistics of random samples with the following density function

(3.1.1.) $$f_{\mu,\sigma}(x) = \exp(-(x-\mu)/\sigma)/\sigma, \qquad (x \geqq \mu).$$

Let us put that

(3.2.1) $$\sigma^{\psi} = c[(1-n+r_1)x_{(r_1+1)}+x_{(r_1+2)}+\cdots+x_{(n-r_2+1)}+(r_2+1)x_{(n-r_2)}],$$

where

(3.1.3) $$c = (n-r_1-r_2-1)^{-1},$$

and where $x_{(r_1+1)}, \cdots, x_{(n-r_2)}$ is a subset of $x_{(1)}, \cdots, x_{(n)}$. It was then proved by Sarhan and Greenberg [14] that σ^{ψ} is a best linear unbiased estimate of σ.

Let us assume k experimental populations with scale parameter $\sigma_i (i=1, 2, \cdots, k)$ and one control population with the parameter σ_0.

Procedure R_3: *If $\sigma_i^{\psi} \geqq \sigma_0^{\psi}$, then we decide that the population Π_i belongs to the group S.*

Now the joint density function of $y_{r_1+1}, \cdots, y_{n-r_2}$ is given by

(3.1.4) $$\xi(y_{r_1+1}, \cdots, y_{n-r_2}) = \frac{n!}{r_1!\, r_2!}[F_{\mu,\sigma}(y_{r_1+1})]^{r_1}[1-F_{\mu,\sigma}(y_{n-r_2})]^{r_2}\prod_{i=r_1+1}^{n-r_2} f_{\mu,\sigma}(y_i),$$

where $y_j = x_{(j)}, j=r_1+1, \cdots, n-r_2$, and

(3.1.5) $$F_{\mu,\sigma}(y) = \int_{\mu}^{y} f_{\mu,\sigma}(x)dx.$$

By means of the transformation:

(3.1.6) $$\begin{cases} (y_i-\mu)/\sigma = s_i, \quad i = r_1+1, \cdots, n-r_2, \\ (x-\mu)/\sigma = s, \\ (t-\mu)/\sigma = t, \end{cases}$$

the density function of (2.1.4) reduces to

(3.1.7) $$\eta(s_{r_1+1}, \cdots, s_{n-r_2}) = \frac{n!}{r_1!\, r_2!}[H(s_{r_1+1})]^{r_1}[1-H(s_{n-r_2})]^{n-2}\prod_{i=r_1 1}^{n-r_2} h(s_i),$$

where

(3.1.8) $$H(x) = \int_0^x h(s)ds \quad \text{and} \quad h(s) = \exp(-s), \ (s \geqq 0).$$

From (3.1.6) we have

(3.1.9) $$\sigma^{\psi}/c = \mu+\sigma\{(1-n+r_1)s_{r_1+1}+s_{r_1+2}+\cdots+s_{,-r_2-1}+(r_2+1)s_{n-r_2}\}.$$

Since the transformation

(3.1.10) $$\begin{aligned} s_{r_1+1} &= \{-(\sigma^{\psi}/c-\mu)/\sigma+s_{r_1+2}+\cdots+s_{n-r_2-1}+(r_2+1)s_{n-r_2}\}/(n-r_1-1). \\ s_i &= s_i, \quad i = r_1-1, \cdots, n-r_2, \end{aligned}$$

has the following Jacobian:

(3.1.11) $$J=(c\sigma)^{-1},$$

the density function of random variable σ^{ψ} is given by

(3.1.12) $$g(\sigma^{\psi})=\int_0^{\infty}\cdots\int_0^{\infty}\eta(s_{r_1+1},\cdots,s_{n-r_2})(c\sigma)^{-1}\prod_{i=r_1+2}^{n-r_2}ds\,.$$

Again by the transformation

(3.1.13) $$(\sigma^{\psi}/c-\mu)/\sigma=\hat{\sigma}\,,$$

we have

(3.1.14) $$g(\hat{\sigma})=\frac{n!}{r_1!\,r_2!}\int_0^{\infty}\cdots\int_0^{\infty}[H(-\hat{\sigma}+s_{r1+2}+\cdots+s_{n-r2-1}+(r_2+1)s_{n-r_2})]^{r_1}$$
$$[1-H(s_{n-r_2})]^{r_2}\exp(-\sum_{i=r_1+2}^{n-r2}s_i)\prod_{i=r_1+2}^{n-r2}ds_i\ ,$$

and $\sigma_0^*\geqq\alpha_i^*$ reduces to $\hat{\sigma}_0/\theta_i\geqq\hat{\sigma}_i$, $i=1,2,\cdots,k$, respectively, providing that $\mu_i=\mu$, $i=1,2,\cdots,k$. Hence for each $\boldsymbol{\sigma}\in\Omega(t)$ the correct decision under R_3 reduces to

(3.1.15) $$\begin{cases}\hat{\sigma}_0/r_i\geqq\hat{\sigma}_i, & i=1,2,\cdots,t \quad\text{and}\\ \hat{\sigma}_j>\hat{\sigma}_0/\delta_j, & j=t+1,\cdots,n\,.\end{cases}$$

Since $g_i(\hat{\sigma}_i)$ equals to $g(\hat{\sigma}_i)$ with $\hat{\sigma}_i=\sigma$, $i=1,2,\cdots,k$, we have

(3.1.6) $$P\{(CD)\,|\,\Omega(t),R_3\}=\int\cdots\int_{\left\{\substack{\hat{\sigma}_0/r_i\geqq\hat{\sigma}_i,\,i=1,2,\cdots,t;\\ \hat{\sigma}_j>\hat{\sigma}_0/\delta_j,\,j=t+1,\cdots,k}\right\}}\prod_{0=i}^{k}[g_i(\sigma_i)d\sigma_i]\,.$$

Hence by the similar method in Section 2.1., we have the following:

Theorem 3.1. *Under the notations and conditions stated above, we obtain*

(3.1.17) $$\mathop{Inf}_{\left\{\substack{r_i\leqq r^*<1<\delta^*\leqq\delta_j,\,i=1,2,\cdots,t;\\ j=t+1,\cdots,k}\right\}}P\{(CD)\,|\,\Omega(t),R_3\}=P\{(CD)\,|\,\Omega^*(t),R_3\}$$
$$=\int_0^{\infty}[G(\sigma_0/r^*)]^t[1-G(\sigma_0/\delta^*)]^{k-t}g_0(\sigma_0)d\sigma_0\,,$$

where three constants n, r_1, ande r_2 are common for all populations, and where

(3.1.18) $$G(x)=G_i(x)=\int_0^{x}g_i(\hat{\sigma}_i)d\hat{\sigma}_i,\ \ i=1,2,\cdots,k\,.$$

3.2. The location parameter case when $x_{(r_1+1)},\cdots,x_{(n-r_2)}$ are used.

Let $f_{\mu,\sigma}(x)$ be the density function defined by (3.1.1). Let us define that

(3.2.1) $$\mu^{\psi}=\left(1+\frac{n-r_1-1}{ac}\right)x_{(r_1+1)}-\frac{1}{ac}\{x_{(r_1+2)}+\cdots+x_{n-r_2-1)}\}-\frac{r_2+1}{ac}x_{(n-r_2)}\,,$$

where

(3.2.2) $$a=\sum_{i=1}^{r_1+1}(n-i+1)^{-1}\quad\text{and}\quad c=(n-r_1-r_2-1)^{-1}\,,$$

and where $x_{(r_1+1)},\cdots,x_{(n-r_2)}$ is the subset of the ordered sample $x_{(1)}<x_{(2)}<\cdots<x_{(n)}$ from the common density function $f_{\mu,\sigma}(x)$. It was then proved that the statistic μ^{ψ} is the best linear unbiased estimate of μ.

In what follows let us give the joint density functions as follows: Let the transformation be

$$\text{(3.2.3)}\qquad \begin{cases}(y_i-\mu)/\sigma = s_i, & i=r_1+1, \cdots, n-n_2\,,\\ (x-\mu)/\sigma = s \text{ and } (y-\mu)/\sigma = t\,,\end{cases}$$

then from the joint density function of $y_{r_1+1}, \cdots, y_{n-r_2}$ in (3.1.4), we obtain

$$\text{(3.2.4)}\qquad \eta(s_{r_1+1}, \cdots, s_{n-r_2}) = \frac{n!}{r_1!\, r_2!}[H(s_{r_1+1})]^{r_1}[1-H(s_{n-r_2})]^{r_2}\prod_{i=r_1+1}^{n-r_2} h(s_i)\,,$$

where

$$\text{(3.2.5)}\qquad H(x) = \int_0^x h(t)dt, \quad h(t) = \exp(-t), t \geqq 0\,.$$

Also the relation (3.2.1) reduces to

$$\text{(3.2.6)}\qquad \mu^{\psi} = \mu+\sigma\{(n-r_1-1+ac)s_{r_1+1}-s_{r_1+2}-\cdots-s_{n-r_2-1}-(r_2+1)s_{n-r_2}\}/(ac)\,.$$

Under the following transformation

$$\text{(3.2.7)}\qquad ac(\mu_i^{\psi}-\mu_i)/\sigma = \hat{\mu}_i \cdots, \quad i=0, 1, \cdots, k\,,$$

we have

$$\text{(3.2.8)}\qquad g_i(\hat{\mu}_i) = e^{-\mu_i}\int_0^\infty \cdots \int_0^\infty \frac{n!}{r_1!\, r_2!}[H(s_{r_1+1})]^r[1-H(s1_{n-r_2})^{r_2}$$
$$\cdot\exp[-ac+n-r_1)s_{r_1+1}-2s_{r_1+3}-\cdots-2s_{n-r_2-1}-r_2 s_{n-r_2}]$$
$$\cdot ds_{r_1+1}ds_{r_1+3}\cdots ds_{n-r_2}\,.$$

From (3.2.7) $\mu_i^{\psi} \geqq \mu_0^{\psi}$ in R_4 reduces to

$$\text{(3.2.9)}\qquad \hat{\mu}_i \geqq ac(\mu_0-\mu_i)/\sigma+\hat{\mu}_0, \quad i=1, 2, \cdots, k\,.$$

Hence we have

$$\text{(3.2.10)}\qquad P\{(CD)\,|\,\Omega(t), R_4\}$$
$$= \int_0^\infty \prod_{i=1}^t [1-G_i(\hat{\mu}_0+ac(\hat{\mu}_0-\mu_i)/\tau)]\prod_{j=t+1}^k [G_j(\hat{\mu}_0+ac(\mu_0-\mu_i)/\sigma)]\cdot g_0(\hat{\mu}_0)d\hat{\mu}_0\,,$$

where

$$\text{(3.2.11)}\qquad G_i(x) = \int_0^x g_i(t)dt$$

and where $g_i(t)$ defined by (3.2.8). However (3.2.12) $\mu_0-\mu_i=-r_i$, $\mu_0-\mu_j=\delta_j$, $i=1, 2, \cdots, t; j=t+1, \cdots, k$, for each element in $\Omega(t)$, we have

$$\text{(3.2.13)}\qquad P\{(CD)\,|\,\Omega(t), R_4\} = \int_0^\infty \prod_{i=1}^t\left[1-G_i\left(\hat{\mu}_0-\frac{acr_i}{\sigma}\right)\right]\cdot\prod_{j=t+1}^k\left[G_j\left(\hat{\mu}_0+\frac{ac\delta_j}{\sigma}\right)\right]g_0(\hat{\mu}_0)d\hat{\mu}_0\,.$$

Therfore we have the following:

Theorem 3.2. *Using the notations and conditions stated above we have*

$$\text{(3.2.14)}\qquad \underset{\substack{\{0<r^*\leqq r_i, i=1,2,\cdots,t,\\ 0<\delta^*\leqq\delta_j, j=t+1,\cdots,k\}}}{Inf} P\{(CD)\,|\,\Omega(t), R_4\} = P\{(CD)\,|\,\Omega^*(t), R_4\}$$
$$= \int_0^\infty\left[1-G\left(\hat{\mu}_0-\frac{acr^*}{\sigma}\right)\right]^t\left[G\left(\hat{\mu}_0+\frac{ac\delta^*}{\sigma}\right)^{k-1} g_0(\hat{\mu}_0)d\hat{\mu}_0\,,$$

where

$$G(x) = \int_0^x g(t)dt, \quad g(t)=g_i(t), \quad i = 1, 2, \cdots, k. \tag{3.2.15}$$

4. Uniform distributions.

Let $\{X_{i1}, X_{i2}, \cdots, X_{in}\}$ be a set of k independent random variables whose joint density function is given by

$$f_i(x_{i1}, x_{i2}, \cdots, x_{in}) = \prod_{j=1}^{n} f_i(x_{ij}), \tag{4.1}$$

where

$$f_i(x_{ij}) = 1/a_i, \quad a_i > x > 0, \quad j = 1, 2, \cdots, n; i = 1, 2, \cdots, k. \tag{4.2}$$

Let y_i be the maximum statistic among $\{x_{i1}, x_{i2}, \cdots, x_{in}\}$, i.e.,

$$Y_i = \max_{1 \leq j \leq n} X_{ij}, \quad i = 1, 2, \cdots, k. \tag{4.3}$$

Then the density function of random variable Y_i be given by

$$\begin{aligned} g_i(y_i) &= n[F_i(y_i)^{n-1}/a_i \\ &= n(y_i/a_i)^{n-1}/a_i, \quad a_i > y_i > 0, \end{aligned} \tag{4.4}$$

where

$$F_i(y_i) = \int_0^{y_i} f_i(t)dt = y_i/a_i, \quad i = 1, 2, \cdots, k. \tag{4.5}$$

Since

$$E\{Y_i\} = \int_0^{a_i} tf_i(t)dt = na/(n+1), \tag{4.6}$$

the random variable

$$Z_i = (n+1)Y_i/n \tag{4.7}$$

is an unbiased estimate of parameter a_i, $i=1, 2, \cdots, k$, respectively. Then the density function of Z_i be given by

$$\begin{aligned} h_i(z_i) &= g(nz_i/(n+1)n)/(n+1) \\ &= n(z_i/b_i)^{n-1}/b_i, \quad b_i > z_i > 0, \end{aligned} \tag{4.8}$$

where

$$b_i = (n+1)a_i/n, \tag{4.9}$$

$i=1, 2, \cdots, k$, respectively. Therefore the joint density function of $z_1, z_2, \cdots, z_k$ is represented by

$$h(z_1, z_2, \cdots, z_k) = \prod_{i=1}^{k} [n(z_i/b_i)^{n-1}/b_i], \quad b_i > z_i > 0, \tag{4.10}$$

$i=1, 2, \cdots, k$. Without loss of generality, let us assume that

$$b_k > b_{k-1} > \cdots > b_1 > 0. \tag{4.11}$$

Our main object in this section is to present the selection procedure which selects the population whose parameter is the minimum value among $\{b_i\}$. In order to carry out this object firstly we give the following notations:

Notation I: Let us put that

$$K_j = \{j+1, j+2, \cdots, k\}, \quad j = 0, 1, 2, \cdots, k-1, \tag{4.12}$$

$$A[(i_1, i_2, \cdots, i_k)] = \{[(x_1, x_2, \cdots, x_k): 0<x_{i_1}<\cdots<x_{i_k}<a_1\}, \tag{4.13}$$

where $(i_1, i_2, \cdots, i_k)$ is a permutation of set $(1, 2, \cdots, k)$.

$$A[(j), (i_1, i_2, \cdots, i_{k-1}) = \{(x_1, x_2, \cdots, x_k): a_1<x_j<a_2; 0<x_{i_1}<\cdots<x_{i_{k-1}}<a_1\}, \tag{4.14}$$

where $j \in K_1$ and $(i_1, i_2, \cdots, i_k)$ is a permutation of set $(1, 2, \cdots, j-1, j+1, \cdots, k)$. $\cdots$

$$A[(j_1, j_2), (0), (i_1, i_2, \cdots, i_{k-2})] = \{(x_1, x_2, \cdots, x_k): a_2<x_{j_1}<x_{j_2}<a_3; \\ 0<x_{i_1}<\cdots<x_{i_{k-2}}<a_1\}, \tag{4.15}$$

where $(j_1, j_2) \in K_2$ and $(i_1, i_2, \cdots, i_{k-2})$ is a permutation of set $K_3-\{j_1, j_2\}$, $\cdots$

$$A[(k), (0), \cdots, (0), (i_1, i_2, \cdots, i_{k-1})] = \{(x_1, x_2, \cdots, x_k): a_{k-1}<x_k<a_k; \\ 0<x_{i_1}<x_{i_2}<\cdots<x_{i_{k-2}}<a_1\}, \cdots \tag{4.16}$$

$$A[(k), (k-1), \cdots, (1)] = \{(x_1, x_2, \cdots, x_k): a_{j-1}<x_j<a_j, \quad j = 1, 2, \cdots, k\}, \tag{4.17}$$

where $a_0=0$.

Notation II:

$$I(A) = \begin{cases} 1, & \text{if } (x_1, x_2, \cdots, x_k) \in A \\ 0, & \text{if } (x_1, x_2, \cdots, x_k) \notin A. \end{cases} \tag{4.18}$$

Theorem 3.1. *Let $U_1<U_2<\cdots<U_k$ denote the ordered statistics of random variables Z_1, $Z_2, \cdots, Z_k$. Under the notations stated above the joint density function of $U_1, U_2, \cdots, U_k$ is given by*

$$\begin{aligned} k(u_1, u_2, \cdots, u_k) &= \textstyle\sum' I(A[i_1, i_2, \cdots, i_k])h_1(U_{i_1})\cdots h_k(u_{i_k}) \\ &+\textstyle\sum' I(A[(j), (i_1, i_2, \cdots, i_{k-1})])h_1(u_{i_1})h_2(u_j)h_3(u_{i_2})\cdots h_k(u_{i_{k-1}})+\cdots \\ &+\textstyle\sum' I(A[(k), (0), \cdots, (0), (i_1, i_2, \cdots, i_{k-1})])h_1(u_{i_1})\cdots h_{k-1}(u_{i-1})h_k(u_k) \\ &+I(A[(k), (k-1), \cdots, (2), (1)])h_1(u_1)\cdots h_k(u_k), \end{aligned} \tag{4.19}$$

where $a_k>u_k>\cdots>u_1>0$, and $\sum I(A[i_1, i_2, \cdots, i_k])$ means the sum of terms which the ordered set $[i_1, i_2, \cdots, i_k]$ over all permutations of $(1, 2, \cdots, k)$, and so on.

Example 1. Let X_1, X_2 be two independent random variables whose repective density function is

$$f_i(x_i) = 1/a_i, \quad a_i>x_i>0, \quad i = i, 2. \tag{4.20}$$

Without loss of generality let us assume that

$$a_2>a_1>0. \tag{4.21}$$

Let $Y_2>Y_1$ be the ordered statistics of X_1 and X_2 and denote that

$$
(4.22)\qquad
\begin{aligned}
A &= \{0<y_1<a_1,\ a_1<y_2<a_2\}\\
B &= \{0<y_1<y_2<a_1\}\\
A' &= \{0<x_1<a_1<x_2<a_2\}\\
B' &= \{0<x_1<x_2<a_1\}\\
B'' &= \{0<x_2<x_1<a_1\}\ ,
\end{aligned}
$$

then each element among A' corresponds to one and only element which belongs to A. Simirally two sets B' and B'' correspond to B. Therefore the joint density function of Y_1 and Y_2 is represented by

$$
(4.23)\qquad
\begin{aligned}
g(y_1, y_2) &= I(A)f_1(y_1)f_2(y_2)+I(B)f_1(y_1)f_2(y_2)+f_1(y_2)f_2(y_1)\}\\
&= \{I(A)+2I(B)\}/(a_1a_2), \qquad a_2>y_2>y_1>0\ .
\end{aligned}
$$

Lemma 1. *The family of density functions $h_b(z)$, $(b>0)$ has monotone likelihood ratio and is stochastically increasing, where*

$$
(4.24)\qquad h_b(z) = n(z/b)^{n-1}/b, \qquad b>z>0\ .
$$

The proof of this lemma follows directly from the definitions by Lehmann [16].

Since the assumption (3.11) is unknown for us, in order to select the population whose parameter equals to $\min_{1\leq i\leq k}\{a_i\}$, let us state the following prepartions:

Notation III:

$$
(4.25)\qquad
\begin{aligned}
\boldsymbol{A} &= \{\boldsymbol{a} = (a_1, a_2, \cdots, a_k)\,|\,0<a_1<\cdots<a_k\}\\
\boldsymbol{A}(\varDelta) &= \{\boldsymbol{a}\,|\,\boldsymbol{a}\in \boldsymbol{A},\ a_i = a_1+\varDelta_i,\ i = 2, 3, \cdots, k\}\\
\boldsymbol{A}_*(\varDelta_*) &= \{\boldsymbol{a}\,|\,\boldsymbol{a}\in \boldsymbol{A},\ a_i = a_1+\varDelta_* = a_*,\ (say),\ i = 2, 3, \cdots, k\}\ ,
\end{aligned}
$$

where $\varDelta_*$ is a positive constant such that

$$
(4.26)\qquad \varDelta_*\leqq \varDelta_i\ (i = 2, 3, \cdots, k)\ .
$$

For every $\boldsymbol{a}\in \boldsymbol{A}(\varDelta)$, let us determine our selection procedure which based upon z_1, z_2, $\cdots$, z_k as follows:

Procedure S: *Decide that*

$a_\nu = \min_{1\leq i\leq k} a_i$ *for the integer* ν *which corresponds to* $\min_{1\leq i\leq k} z_i = z_\nu$.

Notation IV: Let $P\{(CD)\,|\,S,\ \boldsymbol{A}(\varDelta)\}$ be such probability that for every $\boldsymbol{a}\in \boldsymbol{A}(\varDelta)$ we decide that $a_1 = min\{a_i\}$ using the statistical procedure S.

Lemma 2. *For* $b_i\geqq b_*>0$, $i=1, 2, \cdots, k$, we have

$$
(4.27)\qquad
\begin{aligned}
&\sum_{j=1}^{k-1}(-1)^j \sum_{(i_1\cdots i_j)}{}' \left(\frac{b_1{}^j}{b_{i_1}b_{i_2}\cdots b_{i_j}}\right)^n \frac{1}{j+1}\\
&\geqq \sum_{j=1}^{k-1}(-1)^j \binom{k-1}{j}\left(\frac{b_1}{b_k}\right)^{nj}\frac{1}{j+1}\ ,
\end{aligned}
$$

where $\sum'_{(i_1, i_2, \cdots, i_j)\in K_0}$ *means that the ordered set* $(i_1, i_2, \ldots, i_j)$ *run over all permutations of* $(j, j+1, \cdots, k)$.

Proof. We have

(4.28)
$$\int_0^1 \left[\prod_{i=2}^{k}\left\{1-\left(\frac{b_1}{b_k}\right)^n v_1^n\right\}\right] n v_1^{n-1} dv_1$$
$$\geqq \int_0^1 \left(1-\left(\frac{b_1}{b_*}\right)^{k-1} n\, v^{n-1} dv\right.$$
$$= \sum_{h=0}^{k-1} (-1)^h \binom{k-1}{b} \left(\frac{b_1}{b_*}\right)^{nh} /(h+1)\,.$$

For this result we have the following:

Lemma 3. *We have*

(4.29)
$$P\{(CD)\,|\,S,\ A(\varDelta)\} = \sum_{h=0}^{k-1} (-1)^h \sum_{(i_1\cdots i_h)}' \left(\frac{b_1^h}{b_{i_1} b_{i_2}\cdots b_{i_h}}\right)^n /(h+1)\,.$$

The proof of this lemma follows directly.

Theorem 4.2. *Under the notations* I, II, III *and* IV *we have*

(4.31)
$$\underset{\substack{\{\varDelta_* \leqq \varDelta_i \\ i=2,3,\cdots,k\}}}{Inf}\ P\{(CD)\,|\,S,\ A(\varDelta)\} = P\{(CD)\,|\,S,\ A_*(\varDelta_*)\}\,.$$

The proof of this theorem follows from lemmas 1, 2, 3 and 4 directly.

Example 2. In the case when $k=3$, lemma 3 and lemma 4 reduce to

(4.32)
$$P(n,\ (b_1,\ b_2,\ b_3)) \geqq P(n,\ b_*) \text{ for every } n \geqq 2\,,$$

where we have put that

(4.33)
$$P(n,\ (b_1,\ b_2,\ b_3)) = 1-\{(b_1/b_2)^n+(b_1/b_3)^n\}/2+(b_1^2/b_2 b_3)^n/3\,.$$
$$P(n,\ b_*) = 1-(b_1/b_*)^n+(b_1/b_*)^{2n}/3\,.$$

Let us denote the relative efficiencies $P(n, b_*)$ with respect to $P(n, (b_1, b_2, b_3))$ by

(4.34)
$$R(n;\ Q_0,\ Q_i) = P(n,\ Q_0)/P(n,\ Q_i)\,,$$

where $Q_i = C_i,\ D_i,\ i=0, 1, 2, 3$, or E_0, where we have put that

(4.35)
$$C_0 = \{(b_1,\ b_2,\ b_3) = (2,\ 3,\ 3)\},\quad E_0 = \{(2,\ 4,\ 4)\},\quad c_1 = \{(2,\ 3,\ 4)\},$$
$$C_2 = \{(2,\ 3,\ 6)\},\quad C_3 = \{(2,\ 4,\ 8)\},\quad D_0 = (3,\ 4,\ 4),\quad D_1 = \{(3,\ 4,\ 6)\},$$
$$D_2 = \{(3,\ 4,\ 8)\},\quad D_3 = C_3\,.$$

We can appreciate a sort of performance of theorem 4.2 by means of the following table.

Table The relative efficiencies

n	2	3	4	5	10	20
$R(n;\ C_0,\ C_1)$	.9008	.9143	.9330	.9502	.9919	.9999
$R(n;\ C_0,\ C_2)$	.8412	.8757	.9103	.9776	.9914	.9999
$R(n;\ C_0,\ C_3)$	.7320	.7879	.8418	.8884	.9833	.9999
$R(n;\ C_0',\ C_3)$	.9080	.9461	.9691	.9849	.9995	.9999
$R(n;\ D_0,\ D_1)$	.8476	.7839	.8382	.8789	.9725	.9984
$R(n;\ D_0,\ D_2)$	.8046	.7075	.8212	.8688	.9721	.9984
$R(n;\ D_0,\ D_3)$	.6396	.6270	.7070	.7755	.9451	.9968

$R(n;\ C_0,\ C_3) = R(n;\ C_0,\ D_3)$

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