SOME NORM IDEALS RELATED TO THE CLASSICAL SYMMETRIC NORMING FUNCTIONS

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Introduction

This paper is concerned with the theory of norm ideals of the ring $\mathfrak{B}$ of all bounded linear operators on a fixed separable infinite dimensional complex Hilbert space $\mathfrak{H}$. The basic conceptions and results in this theory can be found in the books by Schatten [8] and Gohberg & Kreĭn [2].

Macaev [4] and Gohberg & Kreĭn [1] introduced in 1961 a certain class of non-separable norm ideals. In our previous paper [9], we extended their results to the case of every symmetric norming function and studied the relative conjugate ideals which were introduced by Salinas [7].

Our main purpose in this note is to sharpen our results in the previous paper [9] about the case of the classical symmetric norming functions $\Phi_p(\xi)=(\sum_j \xi_j^p)^{1/p}$ $(1\leqq p<\infty)$, and moreover to extend the class of norm ideals of intermediate type which was obtained by Mitjagin [6] to our case.

This note is the beginning of our investigation of norm ideals related to $\Phi_p(\xi)$. Therefore we shall state in rather detail the preliminary facts for the future. In this paper we shall use the different notations from ones in [9].

In section 1 we shall investigate the notions of adjointness and reflexivility relative to $\Phi_p(\xi)$. In section 2 we shall construct the non-separable norm ideals (=the full norm ideals of maximal type) related to $\Phi_p(\xi)$, and we shall discuss their Φ_p-adjoint. Then incidentally we establish a characterization (Lemma 2.1) of the classical symmetric norming functions $\Phi_p(\xi)$ $(1\leqq p<\infty)$ as the extension of a lemma of Kuroda [3]. In section 3, after Mitjagin [6], we shall obtain new intermediate ideals with respect to our non-separable norm ideals.

1. Φ_p-adjointness and Φ_p-reflexivility

Throughout this paper $\mathfrak{B}$ will denote the ring of all [bounded linear] operators on a fixed separable infinite dimenstional complex Hilbert space $\mathfrak{H}$. And let $\mathbf{k}_o$ be the set of all non-increasing sequences of non-negative real numbers which tend to zero. We denote by $\hat{\mathbf{k}}$ the subset of $\mathbf{k}_o$ consisting of all sequences with a finite number of non-zero terms. In what follows, the elements of $\mathbf{k}_o$ will be denoted by ξ, η, ζ, etc. and the terms of the sequence ξ by ξ_j, $j=1, 2,\cdots$. Given ξ, $\eta\in\mathbf{k}_o$ and non-negative real number a we denote by $\xi+\eta$, $a\xi$, and $\xi\eta$ the following sequences respectively:

$$\xi+\eta=\{\xi_j+\eta_j\},\ a\xi=\{a\xi_j\},\ \text{and}\ \xi\eta=\{\xi_j\eta_j\}.$$

Then obviously $\xi+\eta$, $a\xi$, and $\xi\eta$ are elements of $\mathbf{k}_o$. And the *k-th section* of each sequence $\xi \in \mathbf{k}_o$ will be denoted by $\sigma_k(\xi)$ (i.e., $\sigma_k(\xi) = \{\xi_1, \xi_2, \cdots, \xi_k, 0, 0, \cdots\}$), and the *k-th tail* of ξ by $\tau_k(\xi)$ (i.e., $\tau_k(\xi) = \{\xi_{k+1}, \xi_{k+2}, \cdots\}$).

DEFINITION 1.1. A function $\Phi(\xi)$, defined on $\hat{\mathbf{k}}$, is called a *symmetric norming function* if it has the following properties:

i) $\Phi(\xi) > 0 \quad (\xi \in \hat{\mathbf{k}},\ \xi_1 > 0)$;

ii) $\Phi(\xi + \eta) \leqq \Phi(\xi) + \Phi(\eta) \quad (\xi, \eta \in \hat{\mathbf{k}})$;

iii) for any non-negative real number a

$\Phi(a\xi) = a\Phi(\xi) \quad (\xi \in \hat{\mathbf{k}})$;

iv) $\Phi(\omega) = 1$, where $\omega = \{1, 0, 0, \cdots\} \in \hat{\mathbf{k}}$;

v) if $\xi, \eta \in \hat{\mathbf{k}}$ and

$$\sum_{j=1}^{k} \xi_j \leqq \sum_{j=1}^{k} \eta_j \qquad (k = 1, 2, \cdots),$$

then

$$\Phi(\xi) \leqq \Phi(\eta).$$

Let $\Phi(\xi)$ be any symmetric norming function. If a function $\Psi(\xi)$, defined on $\hat{\mathbf{k}}$, has the above mentioned properties i)~iv) and instead of the property v), has the following:

v′) $\Psi(\xi) \leqq \Psi(\eta)$ whenever $\Phi[\sigma_k(\xi)] \leqq \Phi[\sigma_k(\eta)]$ $(k = 1, 2, \cdots;\ \xi, \eta \in \hat{\mathbf{k}})$,

then we shall say that $\Psi(\xi)$ is a *Φ-symmetric norming function*. Obviously any Φ-symmetric norming function is a symmetric norming function (cf. [9, Lemma 2.8]).

The well-known classical symmetric norming functions $\Phi_p(\xi)$ $(1 \leqq p \leqq \infty)$ are defined by the following:

(1.1) $\quad \Phi_p(\xi) = (\sum_j \xi_j{}^p)^{1/p} \qquad (1 \leqq p < \infty;\ \xi \in \hat{\mathbf{k}})$;

(1.2) $\quad \Phi_\infty(\xi) = \xi_1 \qquad (\xi \in \hat{\mathbf{k}})$.

Then any symmetric norming function $\Phi(\xi)$ satisfies the following inequalities:

(1.3) $\quad \Phi_\infty(\xi) \leqq \Phi(\xi) \leqq \Phi_1(\xi) \qquad (\xi \in \hat{\mathbf{k}})$.

Thus the functions $\Phi_\infty(\xi)$ and $\Phi_1(\xi)$ are respectively called minimal and maximal symmetric norming functions. And by definition our notion of Φ_1-symmetric norming function is the very same one as the ordinary symmetric norming function.

Now we denote by $\mathfrak{S}(\infty)$ the proper largest two-sided ideal of the ring $\mathfrak{B}$, i.e., the set of all compact operators. Following the standard notation, we shall denote by $s(A)$ the sequence of singular numbers (briefly, s-numbers) of the operator $A \in \mathfrak{B}$. If X is in $\mathfrak{S}(\infty)$, then $s(X) = \{s_j(X)\}$ is the sequence of eigenvalues of the operator $(X^*X)^{1/2}$, counted according to multiplicity and arranged in non-increasing order of magnitude. Hence it is clear that $s(X)$ is in $\mathbf{k}_o$.

DEFINITION 1.2. A functional $|X|_s$, defined on a two-sided ideal $\mathfrak{S}$ of the ring $\mathfrak{B}$, is called a *symmetric norm* on $\mathfrak{S}$ if it satisfies the following conditions:

1) $\quad |X|_s > 0 \qquad (X \in \mathfrak{S},\ X \neq O)$;

2) $\quad |cX|_s = |c|\,|X|_s \qquad (X \in \mathfrak{S})$, where c is any complex number;

3) $|X+Y|_s \leqq |X|_s + |Y|_s \quad (X,Y \in \mathfrak{S})$;

4) $|AXB|_s \leqq \|A\|\,|X|_s\|B\| \quad (A,B \in \mathfrak{B},\ X \in \mathfrak{S})$;

5) for any one-dimensional operator $X = (\cdot,f)g\ (f,g \in \mathfrak{H})$,

$$|X|_s = \|X\| = s_1(X) = \|f\|\cdot\|g\|.$$

A two-sided ideal $\mathfrak{S}$ of the ring $\mathfrak{B}$ is called a *norm ideal*, if it is complete with respect to a symmetric norm $|X|_s$ defined on it.

Given a norm ideal $\mathfrak{S}$ with the symmetric norm $|X|_s$, then for any finite dimensional operator K we put

$$(1.4) \qquad \Phi[s(K)] = |K|_s.$$

Obviously the function $\Phi(\xi)$, defined on $\hat{\mathfrak{k}}$, is a symmetric norming function. Conversely, given a symmetric norming function $\Phi(\xi)$, then the set

$$(1.5) \qquad \mathfrak{S}(\Phi) = \{X \in \mathfrak{S}(\infty) : \lim_{n\to\infty} \Phi(\sigma_n[s(X)]) < \infty\}$$

is a norm ideal with the symmetric norm

$$(1.6) \qquad |X|_\Phi = \lim_{n\to\infty} \Phi(\sigma_n[s(X)]) \qquad (X \in \mathfrak{S}(\Phi)).$$

Especially we shall denote by

$$\mathfrak{S}(p) \quad \text{and} \quad |X|_p \qquad (1 \leqq p \leqq \infty)$$

the norm ideal and its symmetric norm corresponding to the symmetric norming function $\Phi_p(\xi)$ defined by (1.1) and (1.2).

DEFINITION 1.3. Let $\mathfrak{S}$ be a norm ideal with the symmetric norm $|X|_s$ and let $\Phi(\xi)$ be the symmetric norming function defined by (1.4). Then $\mathfrak{S}$ is called a *full norm ideal* if two norm ideals $\mathfrak{S}$ and $\mathfrak{S}(\Phi)$ coincide elementwise. In this case the symmetric norms $|X|_s$ and $|X|_\Phi$ are topologically equivalent.

Let us denote by $\mathfrak{K}$ the set of all finite dimensional operators on $\mathfrak{H}$. Given a norm ideal $\mathfrak{S}$ with the symmetric norm $|X|_s$, then we shall denote by $\mathfrak{S}^o$ the closure of $\mathfrak{K}$ with respect to the norm $|X|_s$. Then $\mathfrak{S}^o$ is the *minimal* norm ideal with the symmetric norm $|X|_s$ and it is separable. Particularly we have

$$\mathfrak{S}(p)^o = \mathfrak{S}(p) \qquad (1 \leqq p < \infty).$$

Suppose that $\Phi(\xi)$ and $\Psi(\xi)$ are two symmetric norming functions. We define $\Phi < \Psi$ if and only if

$$(1.7) \qquad \sup_{\xi\in\hat{\mathfrak{k}},\,\xi_1\neq 0} \{\Phi(\xi)/\Psi(\xi)\} < \infty.$$

And we say that $\Phi(\xi)$ and $\Psi(\xi)$ are equivalent $(\Phi \sim \Psi)$ whenever $\Phi < \Psi$ and $\Psi < \Phi$ (cf. [9, Theorem A′]).

Let $\Phi(\xi)$ be any symmetric norming function and η any fixed sequence of $\hat{\mathfrak{k}}$. Then we consider a function

$$\begin{cases} F(\xi) = \dfrac{1}{\Phi(\xi)}\{\sum_j(\xi_j\eta_j)^p\}^{1/p} & (\xi \in \hat{\mathfrak{k}}, \xi \neq 0\,;\, 1 \leqq p < \infty) \\ F(0) = 0\,. & \end{cases}$$

It is clear that by (1.3)

$$(1.8)\qquad F(\xi) \leqq \frac{1}{\xi_1}\Big\{\sum_j(\xi_j\eta_j)^p\Big\}^{1/p} = \Big\{\sum_j\Big(\frac{\xi_j}{\xi_1}\eta_j\Big)^p\Big\}^{1/p} \leqq (\sum_j\eta_j{}^p)^{1/p}$$

for every $\xi \in \hat{\mathfrak{k}}$ and that

$$F(a\xi) = F(\xi) \qquad (\xi \in \hat{\mathfrak{k}})$$

for any positive real number a. Moreover, if n is the largest index corresponding to a non-zero term of the sequence η, then for any sequence $\xi \in \hat{\mathfrak{k}}$ we have

$$F(\xi) \leqq F[\sigma_n(\xi)].$$

Thus, the fuuction $F(\xi)$ is bounded and assumes its finite supremum. Hence we can state the following definition.

DEFINITION 1.4. Let $\Phi(\xi)$ be a symmetric norming function. Then the function

$$(1.9)\qquad \Phi^*{}_{(p)}(\eta) = \sup_{\xi\in\hat{\mathfrak{k}}}\Big\{\frac{1}{\Phi(\xi)}\sum_j(\xi_j\eta_j)^p\Big\}^{1/p} \qquad (1 \leqq p < \infty)$$

has meaning for all sequences $\eta \in \hat{\mathfrak{k}}$. We shall call this function $\Phi^*{}_{(p)}(\eta)$ on $\hat{\mathfrak{k}}$ the Φ_p-*adjoint* of the symmetric norming function $\Phi(\xi)$.

THEOREM 1.5. *The Φ_p-adjoint $\Phi^*{}_{(p)}(\eta)$ of any symmetric norming function $\Phi(\xi)$ is a Φ_p-symmetric norming function, and still more is a symmetric norming function. Then $\Phi^*{}_{(p)} < \Phi_p$. Hence for any real number q such that $1 \leqq q \leqq p$, the set $\mathfrak{S}(q)$ is included in the set $\mathfrak{S}(\Phi^*{}_{(p)})$.*

PROOF. It is clear that $\Phi^*{}_{(p)}(\eta)$ has the properties i)~iii) of Definition 1.1, if one takes into account Minkowski's inequality. Since $\xi_1 \leqq \Phi(\xi)$ and $\Phi(\omega) = 1$ where $\omega = \{1,0,0,\cdots\}$, then $\Phi^*{}_{(p)}(\omega) = 1$. Hence $\Phi^*{}_{(p)}(\eta)$ has the property iv). Finally, it is easily seen that $\Phi^*{}_{(p)}(\eta)$ satisfies the property v$'$) with respect to $\Phi_p(\xi)$, if one takes into account that

$$\frac{1}{\Phi(\xi)}\Big\{\sum_j(\eta_j\xi_j)^p\Big\}^{1/p} = \frac{1}{\Phi(\xi)}\Big\{\sum_j\Big(\sum_{r=1}^{j}\eta_r{}^p\Big)\Big(\xi_j{}^p - \xi_{j+1}{}^p\Big)\Big\}^{1/p}$$

and $\xi_j \geqq \xi_{j+1}$, i.e., $\xi_j{}^p - \xi_{j+1}{}^p \geqq 0$. Then $\Phi^*{}_{(p)}(\eta)$ is a Φ_p-symmetric norming function. And by (1.8), we have

$$\Phi^*{}_{(p)} < \Phi_p.$$

(Moreover, it is generally known that if $\Psi(\xi)$ is any Φ-symmetric norming function, then $\Psi < \Phi$ (cf. [9, Lemma 2.9]).) Hence the last part of the theorem is clear. Therefore the theorem is proved.

DEFINITION 1.6. If a symmetric norming function $\Phi(\xi)$ satisfies that

$$(1.10)\qquad \Phi^{**}_{(p)}(\xi) = \Phi(\xi) \qquad (\xi \in \hat{\mathfrak{k}}),$$

then we shall say that $\Phi(\xi)$ is *Φ_p-reflexive.*

REMARK 1.7. (1) The following remark will be useful in the sequel. If two symmetric norming functions $\Phi(\xi)$ and $\Psi(\xi)$ satisfy the condition

(1.11) $$\{\sum_j(\xi_j\eta_j)^p\}^{1/p} \leqq \Phi(\xi)\cdot\Psi(\eta) \qquad (\xi,\eta \in \hat{\mathbf{k}};\ 1 \leqq p < \infty),$$

and for any sequence ξ (or η) $\in \hat{\mathbf{k}}$ one can find a sequence η (or ξ) $\in \hat{\mathbf{k}}$ for which equality holds in (1.11), then

$$\Phi(\xi) = \Psi^*_{(p)}(\xi) \qquad (\text{or} \quad \Psi(\eta) = \Phi^*_{(p)}(\eta)).$$

(2) If we define for $p = \infty$

$$\Phi^*_{(\infty)}(\eta) = \sup_{\xi \in \hat{\mathbf{k}}} \{\Phi_\infty(\eta\xi) \,/\, \Phi(\xi)\},$$

then it is easily seen that $\Phi^*_{(\infty)}(\eta) = \Phi_\infty(\eta) = \eta_1$ for every symmetric norming function $\Phi(\xi)$. Hence we shall hereafter consider only p such that $1 \leqq p < \infty$.

We have for instance

(1.12) $$\{\sum_j(\xi_j\eta_j)^p\}^{1/p} \leqq \{\sum_j(\xi_j\eta_1)^p\}^{1/p} = (\sum_j\xi_j{}^p)^{1/p}\cdot\eta_1$$
$$= \Phi_p(\xi)\cdot\Phi_\infty(\eta) \qquad (\xi,\eta \in \hat{\mathbf{k}};\ 1 \leqq p < \infty).$$

And let $\xi = \{\xi_1,\xi_2,\cdots,\xi_n,0,0,\cdots\}$ be any sequence from $\hat{\mathbf{k}}$, then for $\eta = \{\underbrace{1,1,\cdots,1}_{n},0,0,\cdots\} \in \hat{\mathbf{k}}$ we have

$$\{\sum_j(\xi_j\eta_j)^p\}^{1/p} = \left(\sum_{j=1}^{n}\xi_j{}^p\right)^{1/p} = \Phi_p(\xi)\cdot\Phi_\infty(\eta).$$

Thus by Remark 1.7, (1)

(1.13) $$(\Phi_\infty)^*_{(p)}(\xi) = \Phi_p(\xi) \qquad (\xi \in \hat{\mathbf{k}}).$$

On the other hand, given any sequence $\eta = \{\eta_1,\eta_2,\cdots,\eta_n,0,0,\cdots\}$ from $\hat{\mathbf{k}}$, then for $\xi = \{\xi_1,0,0,\cdots\} \in \hat{\mathbf{k}}$ $(\xi_1 \neq 0)$

$$\{\sum_j(\xi_j\eta_j)^p\}^{1/p} = \xi_1\eta_1 = \Phi_p(\xi)\cdot\Phi_\infty(\eta).$$

Hence again by Remark 1.7, (1)

(1.14) $$(\Phi_p)^*_{(p)}(\eta) = \Phi_\infty(\eta) \qquad (\eta \in \hat{\mathbf{k}}).$$

Consequently we obtain the following proposition.

PROPOSITION 1.8. *For any real number p such that $1 \leqq p < \infty$, $\Phi_p(\xi)$ and $\Phi_\infty(\xi)$ are mutually Φ_p-adjoint, that is, they are Φ_p-reflexive.*

In more general, we have

PROPOSITION 1.9. *Let $1 \leqq p < \infty$ and let q be any real number such that $p < q < \infty$. Then $\Phi_q(\xi)$ is Φ_p-reflexive. And for q' such that*

$$\frac{1}{q} + \frac{1}{q'} = \frac{1}{p},$$

we have

(1.15) $(\Phi_q)^*_{(p)}(\xi) = \Phi_{q'}(\xi) \qquad (\xi \in \hat{\mathfrak{k}})$.

PROOF. Since $\frac{1}{q/p} + \frac{1}{q'/p} = 1$, then by virtue of Hölder's inequality,

$$\sum_j (\xi_j \eta_j)^p \leqq (\sum_j \xi_j^q)^{p/q} \cdot (\sum_j \eta_j^{q'})^{p/q'}.$$

Hence

(1.16) $\{\sum_j (\xi_j \eta_j)^p\}^{1/p} \leqq \Phi_q(\xi) \cdot \Phi_{q'}(\eta) \qquad (\xi, \eta \in \hat{\mathfrak{k}})$,

and for any fixed $\xi \in \hat{\mathfrak{k}}$ equality holds in (1.16) if and only if

$$\eta_j = c\xi_j^{q/p-1} \qquad (j = 1,2,\cdots),$$

where c is arbitrary positive constant; and for fixed $\eta \in \hat{\mathfrak{k}}$ equality holds in (1.16) if and only if

$$\xi_j = c\eta_j^{q'/p-1} \qquad (j = 1,2,\cdots).$$

Thus by Remark 1.7, (1), the proposition is proved.

Throughout this paper we shall henceforth denote by $\mathfrak{S}(\Phi)^*_{(p)}$ the full norm ideal $\mathfrak{S}(\Phi^*_{(p)})$ associating with the symmetric norming function $\Phi^*_{(p)}(\xi)$. And we shall call it *Φ_p-adjoint ideal* of $\mathfrak{S}(\Phi)$. (The notation and terminology will be justified by Theorem 1.10.)

The following theorem is the central result of the present section.

THEOREM 1.10. *Let* $1 \leqq p < \infty$, *and let* $\Phi(\xi)$ *be any symmetric norming function. Then*

(1.17) $\mathfrak{S}(\Phi)^*_{(p)} = \{X \in \mathfrak{S}(\infty) : XY \in \mathfrak{S}(p)$ *for every* $Y \in \mathfrak{S}(\Phi)^o\}$.

And for any operator $X \in \mathfrak{S}(\Phi)^*_{(p)}$ *its symmetric norm is given by*

(1.18) $$|X|_{\Phi^*_{(p)}} = \sup_{Y \in \mathfrak{S}(\Phi)^o} \frac{\Phi_p[s(X)s(Y)]}{|Y|_\Phi} = \sup_{Y \in \mathfrak{S}(\Phi)^o} \frac{\Phi_p[s(XY)]}{|Y|_\Phi}.$$

PROOF. Let the Schmidt expansion of an operator $Y \in \mathfrak{S}(\Phi)^o$ have the form

$$Y = \sum_j s_j(Y)(\cdot, \phi_j)\psi_j$$

and let

$$Y_n = \sum_{j=1}^{n} s_j(Y)(\cdot, \phi_j)\psi_j$$

be the n-th partial sum of this expansion (see [2, p. 28]). Then we have

$$\lim_{n\to\infty} |Y_n|_\Phi = \lim_{n\to\infty} \Phi(\sigma_n[s(Y)]) = \Phi[s(Y)] = |Y|_\Phi$$

and

$$(1.19) \qquad \lim_{n\to\infty} |Y - Y_n|_\Phi = \lim_{n\to\infty} \Phi(\tau_n[s(Y)]) = 0$$

(cf. [2]). And XY_n is a finite dimensional operator for any $X \in \mathfrak{S}(\Phi)^*_{(p)}$. Hence by the well-known inequality

$$\sum_{j=1}^{k} s_j(XY_n) \leqq \sum_{j=1}^{k} s_j(X)s_j(Y_n) \qquad (k = 1,2,\cdots)$$

for s-numbers, we have

$$\begin{aligned}\Phi_p[s(XY_n)] &\leqq \Phi_p[s(X)s(Y_n)] \\ &\leqq \Phi[s(Y_n)]\cdot \Phi^*_{(p)}[s(X)] \\ &= \Phi(\sigma_n[s(Y)])\cdot \Phi^*_{(p)}[s(X)] \\ &\leqq \Phi[s(Y)]\cdot \Phi^*_{(p)}[s(X)].\end{aligned}$$

Thus we obtain

$$(1.20) \qquad \Phi_p[s(XY_n)] \leqq \Phi_p[s(X)s(Y_n)] \leqq |Y|_\Phi\cdot|X|_{\Phi^*_{(p)}}.$$

Now by [2, Corollary 2.3] we have

$$\begin{aligned}|s_j(XY_n) - s_j(XY)| &\leqq \|XY_n - XY\| \\ &\leqq \|X\|\cdot\|Y_n - Y\| \\ &\leqq \|X\|\cdot|Y - Y_n|_\Phi \qquad (j = 1,2,\cdots).\end{aligned}$$

Hence by the relation (1.20)

$$(1.21) \qquad \Phi_p[s(XY)] \leqq \Phi_p[s(X)s(Y)] \leqq |X|_{\Phi^*_{(p)}}\cdot|Y|_\Phi$$

for every $X \in \mathfrak{S}(\Phi)^*_{(p)}$ and every $Y \in \mathfrak{S}(\Phi)^o$, if one takes into account (1.19) and the continuity of symmetric norming functions (cf. [2, p. 76]). Then we obtain obviously

$$(1.22) \qquad \mathfrak{S}(\Phi)^*_{(p)} \subset \{X \in \mathfrak{S}(\infty) : XY \in \mathfrak{S}(p) \text{ for every } Y \in \mathfrak{S}(\Phi)^o\}.$$

And moreover (1.21) implies for every $X \in \mathfrak{S}(\Phi)^*_{(p)}$

$$\begin{aligned}(1.23) \qquad \sup_{Y\in\mathfrak{S}(\Phi)^o} \{\Phi_p[s(XY)]/|Y|_\Phi\} &\leqq \sup_{Y\in\mathfrak{S}(\Phi)^o} \{\Phi_p[s(X)s(Y)]/|Y|_\Phi\} \\ &\leqq |X|_{\Phi^*_{(p)}}.\end{aligned}$$

On the other hand, let the Schmidt expansion of any operator $X \in \mathfrak{S}(\Phi)^*_{(p)}$ have the form

$$X = \sum_j s_j(X)(\cdot, \phi_j)\psi_j,$$

and put for any $\xi \in \hat{\mathfrak{k}}$

$$K_\xi = \sum_j \xi_j(\cdot, \psi_j)\phi_j.$$

Then we have

$$XK_\xi = \sum_j s_j(X)\xi_j(\cdot, \psi_j)\psi_j.$$

Hence

$$\Phi_p[s(XK_\xi)] = \{\sum_j (s_j(X)\xi_j)^p\}^{1/p}.$$

And since $|K_\xi|_\Phi = \Phi(\xi)$, then

$$\frac{\Phi_p[s(XK_\xi)]}{|K_\xi|_\Phi} = \frac{1}{\Phi(\xi)} \{\sum_j (s_j(X)\xi_j)^p\}^{1/p}.$$

Thus we have

$$\sup_{\xi \in \hat{\mathfrak{k}}} \frac{\Phi_p[s(XK_\xi)]}{|K_\xi|_\Phi} = \sup_{\xi \in \hat{\mathfrak{k}}} \frac{1}{\Phi(\xi)} \{\sum_j (s_j(X)\xi_j)^p\}^{1/p}$$
$$= \Phi^*_{(p)}[s(X)] = |X|_{\Phi^*_{(p)}}.$$

Hence we have

$$(1.24) \qquad |X|_{\Phi^*_{(p)}} = \sup_{\xi \in \hat{\mathfrak{k}}} \frac{\Phi_p[s(XK_\xi)]}{|K_\xi|_\Phi} \leqq \sup_{Y \in \mathfrak{S}(\Phi)^o} \frac{\Phi_p[s(XY)]}{|Y|_\Phi}.$$

Therefore (1.18) is concluded from (1.23) and (1.24).

Finally it remains only to prove the inverse inclusion relation of (1.22). Suppose that $X \in \mathfrak{S}(\infty)$ is any operator such that $XY \in \mathfrak{S}(p)$ for every $Y \in \mathfrak{S}(\Phi)^o$. Further, let $\{P_n\}$ be a monotonically increasing sequence of finite dimensional projections which tends strongly to the identity operator. Then we consider the family of linear transformations: $Y \longrightarrow P_n XY$ of the Banach space $\mathfrak{S}(\Phi)^o$ into the Banach space $\mathfrak{S}(p)$. Since

$$|P_n XY|_p \leqq |P_n X|_{\Phi^*_{(p)}} \cdot |Y|_\Phi$$

and

$$|P_n XY|_p = \Phi_p[s(P_n XY)] \leqq \Phi_p[s(XY)] = |XY|_p < \infty,$$

then by the resonance theorem (see, for example, [10]) it follows that

$$\sup_{Y \in \mathfrak{S}(\Phi)^o} \frac{|P_n XY|_p}{|Y|_\Phi} \leqq c$$

for some $c > 0$, $n = 1,2,\cdots$, and hence

$$(1.25) \qquad \sup_{Y \in \mathfrak{S}(\Phi)^o} \frac{|XY|_p}{|Y|_\Phi} = \sup_{Y \in \mathfrak{S}(\Phi)^o} \frac{\Phi_p[s(XY)]}{|Y|_\Phi} \leqq c.$$

Now let

$$X = \sum_j s_j(X)(\cdot, \phi_j)\psi_j$$

be the Schmidt expansion of X, and define a finite dimensional operator K_ξ by

$$K_\xi = \sum_j \xi_j(\cdot, \psi_j)\phi_j$$

for any $\xi \in \hat{\mathfrak{k}}$. Then, as in the previous paragraph, we have

$$\sup_{\xi \in \hat{\mathfrak{k}}} \frac{\Phi_p[s(XK_\xi)]}{|K_\xi|_\Phi} = \sup_{\xi \in \hat{\mathfrak{k}}} \frac{1}{\Phi(\xi)} \{\sum_j (s_j(X)\xi_j)^p\}^{1/p}.$$

Consequently, by (1.25)

$$\sup_{\xi \in \hat{\mathbf{k}}} \frac{1}{\Phi(\xi)} \{\sum_j (s_j(X)\xi_j)^p\}^{1/p} \leqq c.$$

This implies that $X \in \mathfrak{S}(\Phi)^*_{(p)}$. Hence we have

$$\{X \in \mathfrak{S}(\infty) : XY \in \mathfrak{S}(p) \text{ for every } Y \in \mathfrak{S}(\Phi)^o\} \subset \mathfrak{S}(\Phi)^*_{(p)}.$$

Therefore the proof of the theorem is completed.

The following theorem is a direct consequence of Theorem 1.10, Propositions 1.8, and 1.9.

THEOREM 1.11. *Let* $1 \leqq p < \infty$ *and let* q *and* q' *be two positive numbers such that* $1/q + 1/q' = 1/p$ *(if* $q = p$ *(or* ∞*), then* $q' = \infty$ *(or* p*)). Then for every* $X \in \mathfrak{S}(q')$

$$|X|_{q'} = \sup_{Y \in \mathfrak{S}(q)} \frac{\Phi_p[s(X)s(Y)]}{|Y|_q} = \sup_{Y \in \mathfrak{S}(q)} \frac{\Phi_p[s(XY)]}{|Y|_q}$$

(1.26)

$$= \sup_{Y \in \mathfrak{S}(q)} \frac{|XY|_p}{|Y|_q}$$

and

(1.27) $\mathfrak{S}(q') = \{X \in \mathfrak{S}(\infty) : XY \in \mathfrak{S}(p)$ *for every* $Y \in \mathfrak{S}(q)\}$.

2. Norm ideals of maximal type and their Φ_p-adjoint

All classical norm ideals $\mathfrak{S}(p)$ ($1 \leqq p \leqq \infty$) are separable (minimal) full norm ideals. An example of non-separable full norm ideal was constructed by Macaev [4] in use of the sequence $\{1/(2n-1)\} \in \mathbf{k}_o$. Gohberg & Kreĭn [1] generalized his construction and obtained a class of non-separable full norm ideals. In the previous paper [9], we extended moreover results in [1], and presented new non-separable full norm ideals.

Let $\Phi(\xi)$ be any symmetric norming function and let $\pi = \{\pi_j\} (\in \mathbf{k}_o)$ be a *Φ-binormalizing sequence*, i.e., a sequence such that

(2.1) $\pi_1 = 1, \lim_{n\to\infty} \Phi[\sigma_n(\pi)] = \infty$, and $\lim_{n\to\infty} \pi_n = 0$.

And put

(2.2) $\Phi_\Pi(\xi) = \sup_n \{\Phi[\sigma_n(\xi)]/\Phi[\sigma_n(\pi)]\} \qquad (\xi \in \hat{\mathbf{k}})$.

Then it is easily seen that $\Phi_\Pi(\xi)$ is a symmetric norming function. We shall denote by $\mathfrak{S}(\Phi ; \Pi)$ the full norm ideal associating with $\Phi_\Pi(\xi)$. Then we have $\mathfrak{S}(\Phi ; \Pi)^o \neq \mathfrak{S}(\Phi; \Pi)$. That is, $\mathfrak{S}(\Phi ; \Pi)$ is non-separable (cf. [9, Lemma 2.5]). The full norm ideal $\mathfrak{S}(\Phi ; \Pi)$ has the maximal elements with respect to the order induced by $\Phi(\xi)$ (cf.

[9, Theorems 2.12 and 2.15]). Thus we shall hereafter call the full norm ideal $\mathfrak{S}(\Phi; \Pi)$ *maximal type*. In the special case of $\Phi(\xi) = \Phi_1(\xi) = \sum_j \xi_j$, our norm ideal of maximal trpe $\mathfrak{S}(\Phi_1; \Pi)$ coincides with the ideal given by Gohberg & Kreĭn [1].

In this section we shall, in more detail than the study in [9], investigate the construction and properties of the norm ideal of maximal type $\mathfrak{S}(\Phi_p; \Pi)$ with respect to $\Phi_p(\xi) = (\sum_j \xi_j^p)^{1/p}$.

We begin our considerations by generalizing Kuroda's Lemma [3]. Our result presents a characterization for every symmetric norming function $\Phi_p(\xi)$ $(1 \leqq p < \infty)$.

LEMMA 2.1. *Let $\Phi(\xi)$ be any Φ_p-symmetric norming function $(1 \leqq p < \infty)$. Then*

$$(2.3) \qquad \sup_{\xi \in \hat{\mathfrak{k}}} \frac{\Phi_p(\xi)}{\Phi(\xi)} = \sup_n \frac{n^{1/p}}{\Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)}.$$

In particular, the Φ_p-symmetric norming function $\Phi(\xi)$ is equivalent to $\Phi_p(\xi)$ if and only if

$$(2.4) \qquad \sup_n \frac{n^{1/p}}{\Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)} < \infty.$$

PROOF. Obviously,

$$(2.5) \qquad \sup_n \frac{n^{1/p}}{\Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)} \leqq \sup_{\xi \in \hat{\mathfrak{k}}} \{\Phi_p(\xi)/\Phi(\xi)\}.$$

Let $\xi = \{\xi_1,\xi_2,\cdots,\xi_n,0,0,\cdots\}$ be an arbitrary sequence in $\hat{\mathfrak{k}}$, and $\eta = \{\eta_j\}$ the sequence defined by

$$\eta_1 = \eta_2 = \cdots = \eta_n = \left\{\frac{1}{n} \sum_{j=1}^n \xi_j^p\right\}^{1/p}, \quad \eta_{n+j} = 0 \quad (j = 1,2,\cdots).$$

Then it is easily seen that

$$\sum_{j=1}^k \eta_j^p \leqq \sum_{j=1}^k \xi_j^p \quad (k = 1,2,\cdots),$$

and consequently

$$\Phi_p[\sigma_k(\eta)] \leqq \Phi_p[\sigma_k(\xi)] \quad (k = 1,2,\cdots).$$

Since $\Phi(\xi)$ is a Φ_p-symmetric norming function, then

$$\Phi(\eta) = \left\{\frac{1}{n} \cdot \sum_{j=1}^n \xi_j^p\right\}^{1/p} \cdot \Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)$$
$$\leqq \Phi(\xi),$$

that is,

$$(2.6) \qquad \Phi_p(\xi)/\,\Phi(\xi) \leqq \frac{n^{1/p}}{\Phi(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)}.$$

Thus the relation (2.3) follows from (2.5) and (2.6).

Now let $\eta = \{\eta_j\}$ be the sequence defined by $\eta_1 = \Phi_p(\xi)$, $\eta_j = 0$ $(j = 2,3,\cdots)$ for any sequence $\xi \in \hat{\mathbf{k}}$. Then

$$\Phi_p[\sigma_k(\xi)] \leqq \Phi_p(\xi) = \Phi_p[\sigma_k(\eta)] \quad (k = 1,2,\cdots).$$

Hence again by the Φ_p-symmetry of $\Phi(\xi)$,

$$\Phi(\xi) \leqq \Phi(\eta) = \Phi_p(\xi).$$

Consequently since for any Φ_p-symmetric norming function $\Phi(\xi)$ one has

$$(2.7) \qquad \Phi_p(\xi)/\Phi(\xi) \geqq 1 \qquad (\xi \in \hat{\mathbf{k}}),$$

by virtue of (2.3) the condition (2.4) is necessary and sufficient for the equivalence of the functions $\Phi(\xi)$ and $\Phi_p(\xi)$. The proof of the lemma is complete.

Throughout this section let p be any fixed real number such that $1 \leqq p < \infty$. For an arbitrary non-increasing sequence $\pi = \{\pi_j\}$ of positive real numbers, with $\pi_1 = 1$, we define the function $\Phi_{p;\Pi}(\xi)$ on $\hat{\mathbf{k}}$ by the following:

$$(2.8) \qquad \Phi_{p;\Pi}(\xi) = \sup_n \{\Phi_p[\sigma_n(\xi)]/\Phi_p[\sigma_n(\pi)]\} \qquad (\xi \in \hat{\mathbf{k}}).$$

It can be proved in an evident way that $\Phi_{p;\Pi}(\xi)$ is a Φ_p-symmetric norming function and still more a symmetric norming function. Hence by the remark in the proof of Theorem 1.5, $\Phi_{p;\Pi} < \Phi_p$.

For the sake of abbreviating the notations, we shall in future denote by $\mathfrak{S}(p\,;\Pi)$ the full norm ideal $\mathfrak{S}(\Phi_{p;\Pi})$ associating with the function $\Phi_{p;\Pi}(\xi)$ and by $|X|_{p;\Pi}$ the symmetric norm in $\mathfrak{S}(p\,;\Pi)$.

LEMMA 2.2. *The function $\Phi_{p;\Pi}(\xi)$ is equivalent to $\Phi_p(\xi)$, if and only if $\pi = \{\pi_j\}$ satisfies that*

$$\Phi_p(\pi) = (\sum_j \pi_j{}^p)^{1/p} < \infty.$$

PROOF. As above mentioned, $\Phi_{p;\Pi}(\xi)$ is a Φ_p-symmetric norming function. Since by virtue of the non-increasingness of $\pi = \{\pi_j\}$, for any positive integer n

$$\frac{1}{m}\sum_{j=1}^{m} \pi_j{}^p \geqq \frac{1}{n}\sum_{j=1}^{n} \pi_j{}^p \qquad (1 \leqq m \leqq n),$$

then

$$\Phi_{p;\Pi}(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots) = \sup_{1\leqq m\leqq n}\left\{\frac{m^{1/p}}{\Phi_p[\sigma_m(\pi)]}\right\}$$

$$= \sup_{1\leqq m\leqq n}\left\{1\Big/\Big(\frac{1}{m}\sum_{j=1}^{m} \pi_j{}^p\Big)^{1/p}\right\}$$

$$= 1\Big/\left(\frac{1}{n}\sum_{j=1}^{n}\pi_j{}^p\right)^{1/p} = \frac{n^{1/p}}{\Phi_p[\sigma_n(\pi)]}.$$

Hence

$$\text{(2.9)} \qquad \frac{n^{1/p}}{\Phi_{p;\,\Pi}(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)} = \Phi_p[\sigma_n(\pi)] \quad (n = 1,2,\cdots).$$

Thus by virtue of Lemma 2.1 if and only if $\lim_{n\to\infty}\Phi_p[\sigma_n(\pi)] = \Phi_p(\pi) < \infty$, then $\Phi_{p;\,\Pi}(\xi)$ is equivalent to $\Phi_p(\xi)$. Therefore the lemma is proved.

LEMMA 2.3. *The function $\Phi_{p;\,\Pi}(\xi)$ is equivalent to the minimal symmetric norming function $\Phi_\infty(\xi)$, if and only if $\lim_{n\to\infty}\pi_n > 0$.*

PROOF. Since by the relation (2.9)

$$\Phi_{p;\,\Pi}(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots) = \frac{1}{\left(\frac{1}{n}\sum_{j=1}^{n}\pi_j{}^p\right)^{1/p}} \quad (n = 1,2,\cdots),$$

then

$$\text{(2.10)} \qquad \sup_n \Phi_{p;\,\Pi}(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots) < \infty$$

if and only if $\lim_{n\to\infty}\pi_n > 0$. And as well-known the relation (2.10) is a necessary and sufficient condition in order that $\Phi_{p;\,\Pi}(\xi)$ be equivalent to the minimal symmetric norming function. Therefore the lemma is proved.

Thus let $\pi = \{\pi_j\}$ be any Φ_p-binormalizing sequence, that is, any non-increasing sequence of positive real numbers such that

$$\text{(2.11)} \qquad \pi_1 = 1,\ \lim_{n\to\infty}\Phi_p[\sigma_n(\pi)] = \Phi_p(\pi) = \infty, \text{ and } \lim_{n\to\infty}\pi_n = 0.$$

Then we consider the function

$$\text{(2.12)} \qquad F(X) = \overline{\lim_{n\to\infty}}\,\{\Phi_p(\sigma_n[s(X)])/\Phi_p[\sigma_n(\pi)]\}$$

on $\mathfrak{S}(p\,;\Pi)$. Since by the definition

$$\Phi_p(\sigma_n[s(X)])/\Phi_p[\sigma_n(\pi)] \leqq \Phi_{p;\,\Pi}[s(X)] = |X|_{p;\,\Pi} \qquad (n = 1,2,\cdots),$$

one has $F(X) \leqq |X|_{p;\,\Pi}$. Thus the function $F(X)$ on the Banach space $\mathfrak{S}(p\,;\Pi)$ is continuous at $X = O$.

LEMMA 2.4. *The function $F(X)$ defined by the relation (2.12) on $\mathfrak{S}(p\,;\Pi)$ has the following properties:*

(i) $F(X) \geqq 0 \quad (X \in \mathfrak{S}(p\,;\Pi))$;

(ii) $F(aX) = |a|F(X)$ *for every complex number a*;

(iii) $F(X+Y) \leqq F(X) + F(Y) \quad (X, Y \in \mathfrak{S}(p\,;\Pi))$;

(iv) $F(AXB) \leqq \|A\|\ \|B\|\, F(X)\ (X \in \mathfrak{S}(p\,;\Pi),\ A,B \in \mathfrak{B})$.

And then $F(X)$ *is continuous at every point of* $\mathfrak{S}(p\,;\Pi)$.

PROOF. The property (i) is clear by the definition. By virtue of the well-known properties $s(aX) = |a|s(X)$, $\sum_{j=1}^{n} s_j(X+Y) \leqq \sum_{j=1}^{n} \{s_j(X) + s_j(Y)\}$ $(n = 1,2,\cdots)$, and $s_j(AXB) \leqq \|A\|\ \|B\| s_j(X)$, the properties (ii), (iii), and (iv) are easily seen. Then $F(X)$ is continuous at every point of $\mathfrak{S}(p\,;\Pi)$, because as above mentioned $F(X)$ is continuous at $X = O$.

LEMMA 2.5. *Let* $F(X)$ *be the function defined by* (2.12) *on* $\mathfrak{S}(p\,;\Pi)$. *Then*

(2.13) $\mathfrak{S}(p\,;\Pi)^o = \text{Ker } F \equiv \{X \in \mathfrak{S}(p\,;\Pi) : F(X) = 0\}$.

PROOF. We denote by $\mathfrak{K}$ the set of all finite dimensional operators on the Hilbert space $\mathfrak{H}$. Then it follows from $\lim_{n\to\infty} \Phi_p[\sigma_n(\pi)] = \infty$ and the relation (2.12) that $F(K) = 0$ for every $K \in \mathfrak{K}$. Hence by virtue of the continuity of $F(X)$, $F(X) = 0$ on the closure $\mathfrak{S}(p\,;\Pi)^o$ of $\mathfrak{K}$ in $\mathfrak{S}(p\,;\Pi)$. Thus we have that $\mathfrak{S}(p\,;\Pi)^o \subset \text{Ker } F$.

On the other hand for every $X \in \mathfrak{S}(p\,;\Pi)$

(2.14) $$\inf_{K\in\mathfrak{K}} |X - K|_{p;\Pi} = \lim_{n\to\infty} \Phi_{p;\Pi}(\tau_n[s(X)]).$$

Again by (2.12) there exists, for an arbitrary positive number ϵ, a positive integer k_0 such that for $k > k_0$

$$\Phi_p\{\sigma_k[s(X)]\}/\Phi_p[\sigma_k(\pi)] < F(X) + \epsilon.$$

Hence still more for $k > k_0$

$$\Phi_p\{\sigma_k(\tau_n[s(X)])\}/\Phi_p[\sigma_k(\pi)] < F(X) + \epsilon.$$

It is easily seen that for sufficiently large n the above relation will also hold for $k \leqq k_0$. Consequently

$$\limsup_{n\to\infty}\ \sup_k \Phi_p\{\sigma_k(\tau_n[s(X)])\}/\Phi_p[\sigma_k(\pi)]$$
$$= \lim_{n\to\infty} \Phi_{p;\Pi}(\tau_n[s(X)]) \leqq F(X) + \epsilon.$$

Hence it follows from the relation (2.14) and the arbitraryness of $\epsilon > 0$ that

$$\inf_{K\in\mathfrak{K}} |X - K|_{p;\Pi} \leqq F(X) \quad \text{for every } X \in \mathfrak{S}(p\,;\Pi).$$

Obviously the relation implies that $\text{Ker } F \subset \mathfrak{S}(p\,;\Pi)^o$.
Therefore the proof of the lemma is complete.

THEOREM 2.6. *Let* $\pi = \{\pi_j\}$ *be any* Φ_p*-binormalizing sequence, i.e., any non-increasing sequence of positive real numbers satisfying* (2.11). *Then the function* $\Phi_{p;\Pi}(\xi)$ *is not equivalent to either the minimal* $\Phi_\infty(\xi)$ *or* $\Phi_p(\xi)$, *and moreover in this* *case* $\mathfrak{S}(p\,;\Pi) \neq \mathfrak{S}(p\,;\Pi)^{o}$, *that is*, $\mathfrak{S}(p\,;\Pi)$ *is a non-separable full norm ideal.*

PROOF. By virtue of Lemmas 2.2 and 2.3 the first part of the theorem is clear. Nothing remains but to show that $\mathfrak{S}(p\,;\Pi) \neq \mathfrak{S}(p\,;\Pi)^{o}$.

Since we have

$$\begin{aligned}\Phi_{p;\Pi}(\pi) &= \lim_{n\to\infty} \Phi_{p;\Pi}[\sigma_n(\pi)] \\ &= \lim_{n\to\infty} \{\sup_{1\leq k\leq n} (\Phi_p[\sigma_k(\sigma_n(\pi))]/\Phi_p[\sigma_k(\pi)])\} \\ &= \lim_{n\to\infty} \{\sup_{1\leq k\leq n} (\Phi_p[\sigma_k(\pi)]/\Phi_p[\sigma_k(\pi)])\} \\ &= 1,\end{aligned}$$

if X_π is any compact operator which has $\pi = \{\pi_j\}$ as its s-numbers, then $X_\pi \in \mathfrak{S}(p\,;\Pi)$.

But it follows from the relation (2.12) that $F(X_\pi) = 1$. Hence by Lemma 2.5, $X_\pi \notin \mathfrak{S}(p\,;\Pi)^{o}$. Therefore the theorem is proved.

Next let $\pi = \{\pi_j\}$ be an arbitrary non-increasing sequence of positive real numbers with $\pi_1 = 1$. Then we shall define the function $\Phi_{p;\pi}(\xi)$ on $\hat{\mathfrak{k}}$ by the following relation:

$$\Phi_{p;\pi}(\xi) = \Phi_p(\pi\xi) \quad (\xi \in \hat{\mathfrak{k}}). \tag{2.15}$$

It is readily to see that the function $\Phi_{p;\pi}(\xi)$ satisfies the conditions i), iii), and iv) of Definition 1.1. Bearing in the mind Minkowski's inequality, we can easily prove the condition ii) of Definition 1.1. And also remarking Abel's lemma:

$$\sum_{j=1}^{n} (\pi_j\xi_j)^p = \pi_n{}^p\Big(\sum_{j=1}^{n} \xi_j{}^p\Big) + \sum_{j=1}^{n-1} (\pi_j{}^p - \pi_{j+1}{}^p)\Big(\sum_{r=1}^{j} \xi_r{}^p\Big)$$

and $\pi_j{}^p - \pi_{j+1}{}^p \geq 0$, it is known that the condition v′) is satisfied by $\Phi_{p;\pi}(\xi)$. Thus the function $\Phi_{p;\pi}(\xi)$ is a Φ_p-symmetric norming function (still more, a symmetric norming function). Thereupon we shall denote by $\mathfrak{S}(p\,;\pi)$ and $|X|_{p;\pi}$ the full norm ideal and its symmetric norm associating with the function $\Phi_{p;\pi}(\xi)$ respectively.

LEMMA 2.7. *If and only if*

$$\lim_{n\to\infty} \Phi_p[\sigma_n(\pi)] = \Big(\sum_{j=1}^{\infty} \pi_j{}^p\Big)^{1/p} < \infty,$$

then the function $\Phi_{p;\pi}(\xi)$ *is equivalent to the minimal one* $\Phi_\infty(\xi)$. *If and only if*

$$\lim_{n\to\infty} \pi_n > 0,$$

then the function $\Phi_{p;\pi}(\xi)$ *is equivalent to the function* $\Phi_p(\xi)$.

PROOF. The first statement of the lemma is accompleshed because $\Phi_{p;\pi}(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots) = \Phi_p[\sigma_n(\pi)]$. Next we have

$$\frac{n^{1/p}}{\Phi_{p;\pi}(\underbrace{1,1,\cdots,1}_{n},0,0,\cdots)} = \frac{1}{\Big\{\dfrac{1}{n}\displaystyle\sum_{j=1}^{n} \pi_j{}^p\Big\}^{1/p}}.$$

Hence by virtue of Lemma 2.1 the second statement of the lemma is also proved.

THEOREM 2.8. *For an arbitrary Φ_p-binormalizing sequence $\pi = \{\pi_j\}$, the function $\Phi_{p;\pi}(\xi)$ is not equivalent either to the minimal one $\Phi_\infty(\xi)$ or the function $\Phi_p(\xi)$, and moreover in this case $\mathfrak{S}_{p;\pi}(\xi)$ is monomormalizing* (cf. [2, p.88]). *Hence $\mathfrak{S}(p\,;\pi)$ is a separable (minimal) full norm ideal.*

PROOF. The first part of this theorem follows from Lemma 2.7. Nothing remains but to prove that the function $\Phi_{p;\pi}(\xi)$ is mononormalizing, i.e.,

(2.16) $\lim_{n\to\infty} \Phi_{p;\pi}(\tau_n[s(X)]) = 0$ for every $X \in \mathfrak{S}(p\,;\pi)$.

By virtue of the relation (2.15) and $X \in \mathfrak{S}(p\,;\pi)$, there exists, for an arbitrary $\epsilon > 0$, a positive integer m such that

$$\sum_{j=m+1}^{\infty} [\pi_j s_j(X)]^p < \frac{\epsilon}{2}.$$

And moreover, since $\lim_{j\to\infty} s_j(X) = 0$, we can find an n such that

$$\sum_{j=}^{m} [\pi_j s_{n+j}(X)]^p < \frac{\epsilon}{2}.$$

Hence

$$\sum_{j=1}^{\infty} [\pi_j s_{n+j}(X)]^p \leqq \sum_{j=1}^{m} [\pi_j s_{n+j}(X)]^p + \sum_{j=m+1}^{\infty} [\pi_j s_j(X)]^p < \epsilon.$$

Thus

$$\Phi_{p;\pi}(\tau_n[s(X)]) = \left\{\sum_{j=1}^{\infty} [\pi_j s_{n+j}(X)]^p\right\}^{1/p} < \epsilon^{1/p}.$$

Consequently by arbitraryness of $\epsilon > 0$ we have

$$\lim_{n\to\infty} \Phi_{p;\pi}(\tau_n[s(X)] = 0.$$

Therefore the theorem is proved.

LEMMA 2.9. *Let $\hat{\mathfrak{k}}_n$ be the cone of real n-dimensional coordinate space, consisting of all vectors $\xi = \{\xi_j\}^n_1$ for which $\xi_1 \geqq \xi_2 \geqq \cdots \geqq \xi_n \geqq 0$, and let $\{\pi_j\}^n_1$ be an arbitrary system of n positive real numbers. Then*

$$\left\{\sum_{j=1}^{n} (\xi_j\eta_j)^p\right\}^{1/p} \leqq \left\{\sum_{j=1}^{n} (\pi_j\xi_j)^p\right\}^{1/p} \cdot \max_{1\leq r\leq n} \left\{\left(\sum_{j=1}^{r} \eta_j^p\right)^{1/p} \Big/ \left(\sum_{j=1}^{r} \pi_j^p\right)^{1/p}\right\} \tag{2.17}$$

for any pair of vectors $\xi = \{\xi_j\}^n_1$, $\eta = \{\eta_j\}^n_1$ in $\hat{\mathfrak{k}}_n$. For every non-zero vector $\xi \in \hat{\mathfrak{k}}_n$ ($\eta \in \hat{\mathfrak{k}}_n$) there exists a non-zero vector $\eta \in \hat{\mathfrak{k}}_n$ ($\xi \in \hat{\mathfrak{k}}_n$) for which equality holds in (2.17).

PROOF. Put $\xi'_j = \xi_j^p$ $(j = 1,2,\cdots,n)$ for every $\xi = \{\xi_j\}^n_1 \in \hat{\mathfrak{k}}_n$. Then $\xi' = \{\xi'_j\}^n_1 \in \hat{\mathfrak{k}}_n$ and the correspondence $\xi \rightarrow \xi'$ is bijective. Accordingly in order to prove this lemma it is sufficient to show that for every $\xi' = \{\xi'_j\}^n_1$, $\eta' = \{\eta'_j\}^n_1 \in \hat{\mathfrak{k}}_n$

$$(2.18) \qquad \sum_{j=1}^{n} \xi'_j \eta'_j \leqq \sum_{j=1}^{n} \pi_j{}^p \xi'_j \cdot \max_{1 \leq r \leq n} \left\{ \sum_{j=1}^{r} \eta'_j / \sum_{j=1}^{r} \pi_j{}^p \right\}$$

and for every non-zero vector $\xi' \in \hat{\mathfrak{k}}_n$ $(\eta' \in \hat{\mathfrak{k}}_n)$ there exists a non-zero vector $\eta' \in \hat{\mathfrak{k}}_n$ $(\xi' \in \hat{\mathfrak{k}}_n)$ for which equality holds in (2.18). But this is just the result in [2, Lemma 15.2].

By virtue of Remark 1.7, (1) and the lemma just proved, the symmetric norming functions $\Phi_{p;\Pi}(\xi)$ and $\Phi_{p;\pi}(\xi)$ are mutually Φ_p-adjoint. Hence by Theorem 1.10 the following theorem is valid.

THEOREM 2.10. *Let* $1 \leqq p < \infty$ *and let* $\pi = \{\pi_j\}$ *be a* Φ_p*-binormalizing sequence. Then for the triple of norm ideals* $\mathfrak{S}(p\,;\Pi)^o$, $\mathfrak{S}(p\,;\pi)$, *and* $\mathfrak{S}(p\,;\Pi)$, *each ideal is the* Φ_p*-adjoint of the preceding one. Namely,*

$$(2.19) \qquad \mathfrak{S}(p\,;\pi) = \{X \in \mathfrak{S}(\infty) : XY \in \mathfrak{S}(p) \textit{ for every } Y \in \mathfrak{S}(p\,;\Pi)^o\},$$

$$(2.20) \qquad |X|_{p;\pi} = \sup_{Y \in \mathfrak{S}(p;\Pi)^o} \frac{\Phi_p[s(X)s(Y)]}{|Y|_{p;\Pi}} = \sup_{Y \in \mathfrak{S}(p;\Pi)^o} \frac{\Phi_p[s(XY)]}{|Y|_{p;\Pi}}\,;$$

$$(2.21) \qquad \mathfrak{S}(p\,;\Pi) = \{X \in \mathfrak{S}(\infty) : XY \in \mathfrak{S}(p) \textit{ for every } Y \in \mathfrak{S}(p\,;\pi)\},$$

$$(2.22) \qquad |X|_{p;\Pi} = \sup_{Y \in \mathfrak{S}(p;\pi)} \frac{\Phi_p[s(X)s(Y)]}{|Y|_{p;\pi}} = \sup_{Y \in \mathfrak{S}(p;\pi)} \frac{\Phi_p[s(XY)]}{|Y|_{p;\pi}}\,.$$

3. Norm ideals of intermediate type

Mitjagin [6] has shown that for any binormalizing sequence $\pi = \{\pi_j\}$ there exists an intermediate norm ideal $\mathfrak{S}(1\,;\Pi,\alpha)$, strictly included between $\mathfrak{S}(1\,;\Pi)^o$ and $\mathfrak{S}(1\,;\Pi)$, and that the symmetric norm in $\mathfrak{S}(1\,;\Pi,\alpha)$ is defined by the symmetric norming function $\Phi_{1;\Pi}(\xi)$.

Throughout this section again let p be an arbitrary fixed real number such that $1 \leqq p < \infty$. Then as the extension of the results in [6], we shall prove that for any Φ_p-binormalizing sequence $\pi = \{\pi_j\}$ there exists a norm ideal of intermediate type $\mathfrak{S}(p\,;\Pi,\alpha)$ with the symmetric norm $|X|_{p;\Pi}$, properly included between the minimal norm ideal $\mathfrak{S}(p\,;\Pi)^o$ and the full norm ideal of maximal type $\mathfrak{S}(p\,;\Pi)$. But our construction will essentially be owing to the idea of Mitjagin [6].

Let $\alpha = \{\alpha_j\}$ be any non-increasing sequence of positive real numbers and let $\pi = \{\pi_j\}$ be a Φ_p-binormalizing sequence. Then we consider a function

$$(3.1) \qquad G(X) = \overline{\lim_{n \to \infty}}\ \{\Phi_p(\sigma_n[s(X)\alpha])/\Phi_p[\sigma_n(\pi\alpha)]\}$$

defined on $\mathfrak{S}(p\,;\,\Pi)$. By the definition

$$\Phi_p(\sigma_n[s(X)]) \leqq |X|_{p;\,\Pi}\cdot\Phi_p[\sigma_n(\pi)] = \Phi_p[\sigma_n(|X|_{p;\,\Pi}\pi)] \quad (n = 1,2,\cdots)$$

for every $X \in \mathfrak{S}(p\,;\,\Pi)$. Hence by virtue of Abel's lemma we have

$$\Phi_p(\sigma_n[s(X)\alpha]) \leqq \Phi_p[\sigma_n(|X|_{p;\,\Pi}\pi\alpha)] = |X|_{p;\,\Pi}\Phi_p[\sigma_n(\pi\alpha)].$$

i.e.,

$$\Phi_p(\sigma_n[s(X)\alpha])/\Phi_p[\sigma_n(\pi\alpha)] \leqq |X|_{p;\,\Pi} \qquad (n = 1,2,\cdots).$$

Thus

$$G(X) \leqq |X|_{p;\,\Pi} \qquad (X \in \mathfrak{S}(p\,;\,\Pi)),$$

and the function $G(X)$ on the Banach space $\mathfrak{S}(p\,;\,\Pi)$ is continuous at $X = O$. Similarly in Lemma 2.4 we have the following lemma.

LEMMA 3.1. *The function* $G(X)$ *defined by the relation* (3.1) *on* $\mathfrak{S}(p\,;\,\Pi)$ *has the following properties*:

(i) $G(X) \geqq 0 \qquad (X \in \mathfrak{S}(p\,;\,\Pi))$;

(ii) $G(aX) = |a|G(X)$ *for every complex number* a;

(iii) $G(X+Y) \leqq G(X) + G(Y) \qquad (X,Y \in \mathfrak{S}(p\,;\,\Pi))$;

(iv) $G(AXB) \leqq \|A\|\,\|B\|\,G(X) \qquad (X \in \mathfrak{S}(p\,;\,\Pi),\ A,B \in \mathfrak{B})$.

And then $G(X)$ *is continuous on* $\mathfrak{S}(p\,;\,\Pi)$.

PROOF. Like the proof of Lemma 2.4, the properties (i) and (ii) are clear. Since $\sum_{j=1}^{n} s_j(X+Y) \leqq \sum_{j=1}^{n} \{s_j(X) + s_j(Y)\}$ $(n = 1,2,\cdots)$ by the property of s-numbers, we have

$$\sum_{j=1}^{n} s_j(X+Y)^p \leqq \sum_{j=1}^{n} \{s_j(X) + s_j(Y)\}^p \quad (n = 1,2,\cdots).$$

Then by virtue of Abel's lemma,

$$\sum_{j=1}^{n} \{s_j(X+Y)\alpha_j\}^p \leqq \sum_{j=1}^{n} \{[s_j(X) + s_j(Y)]\alpha_j\}^p \quad (n=1,2,\cdots).$$

Consequently by Minkowski's inequality,

$$\begin{aligned}\Phi_p(\sigma_n[s(X+Y)\alpha]) &\leqq \Phi_p\{\sigma_n([s(X) + s(Y)]\alpha)\} \\ &\leqq \Phi_p(\sigma_n[s(X)\alpha]) + \Phi_p(\sigma_n[s(Y)\alpha])\end{aligned}$$

$(n = 1,2,\cdots)$. Hence the property (iii) is obtained from the relation (3.1). The property (iv) follows easily from that

$$\Phi_p(\sigma_n[s(AXB)]) \leqq \|A\|\,\|B\|\,\Phi_p(\sigma_n[s(X)]) \qquad (n = 1,2,\cdots)$$

and again Abel's lemma. So the function $G(X)$ is continuous.

LEMMA 3.2. *Let $G(X)$ be the function defined by the relation* (3.1) *on* $\mathfrak{S}(p\,;\Pi)$. *Then the set*

(3.2) $\mathfrak{S}(p\,;\Pi,\alpha) \equiv \text{Ker}\ G \equiv \{X \in \mathfrak{S}(p\,;\Pi) : G(X) = 0\}$

is a norm ideal with the symmetric norm $|X|_{p;\,\Pi}$. *And* $\mathfrak{S}(p\,;\Pi,\alpha) \neq \mathfrak{S}(p\,;\Pi)$.

PROOF. By virtue of the properties (ii), (iii) and (iv) of Lemma 3.1 it is easily to show that the set $\mathfrak{S}(p\,;\Pi,\alpha)$ is an (algebraic) two-sided ideal. On the other hand by the continuity of $G(X)$, the set $\mathfrak{S}(p\,;\Pi,\alpha)$ is a closed subspace of $\mathfrak{S}(p\,;\Pi)$. Hence $\mathfrak{S}(p\,;\Pi,\alpha)$ is a norm ideal with the symmetric norm $|X|_{p;\,\Pi}$.

In the same way in the proof of Theorem 2.6, let X_π be any compact operator with its s-numbers $\pi = \{\pi_j\}$. Then $X_\pi \in \mathfrak{S}(p\,;\Pi)$. But $X_\pi \notin \mathfrak{S}(p\,;\Pi,\alpha)$ because $G(X_\pi) = 1$ by the relation (3.1). Therefore this lemma is proved.

REMARK 3.3. (1) If the non-increasing sequence $\alpha = \{\alpha_j\}$ satisfies $\lim_{n\to\infty} \alpha_n = \delta > 0$, then obviously

$$\alpha_1 \Phi_p(\sigma_n[s(X)]) \geqq \Phi_p(\sigma_n[s(X)\alpha]) \geqq \delta\Phi_p(\sigma_n[s(X)])$$

and

$$\alpha_1 \Phi_p[\sigma_n(\pi)] \geqq \Phi_p[\sigma_n(\pi\alpha)] \geqq \delta\Phi_p[\sigma_n(\pi)].$$

Hence by the relations (2.12) and (3.1)

$$\frac{\alpha_1}{\delta} F(X) \geqq G(X) \geqq \frac{\delta}{\alpha_1} F(X).$$

Thus we have by Lemma 2.5

$$\mathfrak{S}(p\,;\Pi,\alpha) = \mathfrak{S}(p\,;\Pi)^o.$$

(2) If

$$\lim_{n\to\infty} \Phi_p[\sigma_n(\pi\alpha)] = \Big\{\sum_{j=1}^{\infty} (\pi_j\alpha_j)^p\Big\}^{1/p} < \infty,$$

then clearly $\mathfrak{S}(p\,;\Pi,\alpha) = \{O\}$.

(3) If $\lim_{n\to\infty} \Phi_p[\sigma_n(\pi\alpha)] = \infty$, then $G(K) = 0$ for every $K \in \mathfrak{K}$, Hence

$$\mathfrak{S}(p\,;\Pi)^o \subset \mathfrak{S}(p\,;\Pi,\alpha).$$

By virtue of Remark 3.3 we may consider only the non-increasing sequence $\alpha = \{\alpha_j\}$ which satisfies

(3.3) $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \Phi_p[\sigma_n(\pi\alpha)] = \infty$.

LEMMA 3.4. *For sequences* $\pi = \{\pi_j\}$, $\alpha = \{\alpha_j\} \in \mathbf{k}_o$, (3.4) $\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)] \leqq n^{1/p}\Phi_p[\sigma_n(\pi\alpha)] \quad (n = 1,2,\cdots).$

PROOF. It is easily seen that

$$\{\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)]\}^p = \sum_{j=1}^{n}\pi_j{}^p\cdot\sum_{j=1}^{n}\alpha_j{}^p$$
$$\leqq n\sum_{j=1}^{n}(\pi_j\alpha_j)^p = \{n^{1/p}\Phi_p[\sigma_n(\pi\alpha)]\}^p$$

(cf. [6, Lemma 7]). Thus this lemma is proved.

LEMMA 3.5. *Let* $\pi = \{\pi_j\}$, $\alpha = \{\alpha_j\}$ *be sequences of* $\mathbf{k}_0$. *If* $\Phi_p(\pi\alpha) < \infty$, *then*

(3.5) $\lim_{n\to\infty}\{n^{-1/p}\cdot\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)]\} = 0.$

PROOF. For any $\epsilon > 0$ we choose a positive integer n_0 such that

(3.6) $\Phi_p[\tau_n(\pi\alpha)] = \left\{\sum_{j=n+1}^{\infty}(\pi_j\alpha_j)^p\right\}^{1/p} < \epsilon \quad (n \geqq n_0).$

On the other hand

(3.7) $n^{-1/p}\cdot\Phi_p[\sigma_n(\pi)] = \left(\frac{1}{n}\sum_{j=1}^{n}\pi_j{}^p\right)^{1/p} \longrightarrow 0 \quad (n \longrightarrow \infty),$

and

(3.7′) $n^{-1/p}\cdot\Phi_p[\sigma_n(\alpha)] \longrightarrow 0 \quad (n \longrightarrow \infty).$

Moreover for every $n \geqq n_0$

$$\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)]$$
$$\leqq \{\Phi_p[\sigma_{n_0}(\pi)] + \Phi_p(\sigma_{n-n_0}[\tau_{n_0}(\pi)])\}\cdot\{\Phi_p[\sigma_{n_0}(\alpha)] + \Phi_p(\sigma_{n-n_0}[\tau_{n_0}(\alpha)])\}$$
$$\leqq \Phi_p[\sigma_{n_0}(\pi)]\cdot\Phi_p[\sigma_{n_0}(\alpha)] + \Phi_p[\sigma_{n_0}(\pi)]\cdot\Phi_p[\sigma_n(\alpha)]$$
$$+ \Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_{n_0}(\alpha)] + \Phi_p(\sigma_n[\tau_{n_0}(\pi)])\cdot\Phi_p(\sigma_n[\tau_{n_0}(\alpha)]).$$

And by Lemma 3.4 we have for every $n \geqq n_0$

$$\Phi_p(\sigma_n[\tau_{n_0}(\pi)])\cdot\Phi_p(\sigma_n[\tau_{n_0}(\alpha)]) \leqq n^{1/p}\Phi_p(\sigma_n[\tau_{n_0}(\pi\alpha)])$$
$$\leqq n^{1/p}\Phi_p[\tau_{n_0}(\pi\alpha)].$$

Hence by (3.6), (3.7) and (3.7′)

$$\overline{\lim_{n\to\infty}}\{n^{-1/p}\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)]\} \leqq \epsilon.$$

The lemma is thus proved.

Now after the idea of Mitjagin [6] we shall construct the non-increasing sequence $\alpha = \{\alpha_j\}$ which satisfies the relation (3.3) and for which $\mathfrak{S}(p\,;\Pi,\alpha)$ includes $\mathfrak{S}(p\,;\Pi)^0$ properly.

Let $\pi = \{\pi_j\}$ be any Φ_p-binormalizing sequence. Set

(3.8) $Q_n = n/\sum_{j=1}^{n}\pi_j{}^p \quad (n = 1,2,\cdots),$

then $\{Q_n\}$ is a non-decreasing sequence because

$$Q_{n+1} - Q_n = \frac{\sum_{j=1}^{n} (\pi_j{}^p - \pi_{n+1}{}^p)}{\sum_{j=1}^{n+1} \pi_j{}^p \cdot \sum_{j=1}^{n} \pi_j{}^p} \geqq 0 .$$

Moreover we have

$$(3.9) \qquad \lim_{n\to\infty} Q_n = \lim_{n\to\infty} \frac{1}{\frac{1}{n}\sum_{j=1}^{n} \pi_j{}^p} = \frac{1}{\lim_{n\to\infty} \pi_n{}^p} = \infty$$

and

$$(3.10) \qquad \lim_{n\to\infty} \frac{Q_n}{n} = \lim_{n\to\infty} \frac{1}{\sum_{j=1}^{n} \pi_j{}^p} = 0 .$$

Now let $\{Q_n'\}$ be a *convex regularization* of the sequence $\{Q_n\}$ (cf. [5] or [6]). Then the sequence $\{Q_n'\}$ has the following properties :

(3.11) $Q_n' \leqq Q_n \quad (n = 1,2,\cdots)$;

(3.12) $Q_{n+1}' - Q_n' \leqq Q_n' - Q_{n-1}' \quad (n = 2,3,\cdots)$;

(3.13) $Q_{n_k}' = Q_{n_k}$ for an infinite number of indices $n_k \quad (k = 1,2,\cdots)$.

And set

(3.14) $\alpha_1 = Q_1'^{1/p}, \ \alpha_j = (Q_j' - Q_{j-1}')^{1/p} \quad (j = 2,3,\cdots)$,

then by (3.12) the sequence $\alpha = \{\alpha_j\}$ is a non-increasing one of positive real numbers. Moreover by (3.11) and (3.14)

$$(3.15) \qquad \sum_{j=1}^{n} \alpha_j{}^p = Q_n' \leqq Q_n \quad (n = 1,2,\cdots).$$

Thus by (3.10)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \alpha_j{}^p \leqq \lim_{n\to\infty} \frac{Q_n}{n} = 0 ,$$

and so this implies that $\lim_{n\to\infty} \alpha_n = 0$.

Such a sequence $\alpha = \{\alpha_j\}$ will be called a *sequence constructed by Mitjagin's method from the* Φ_p*-binormalizing sequence* $\pi = \{\pi_j\}$.

This $\alpha = \{\alpha_j\}$ is our desirous one.

LEMMA 3.6. *Let* $\alpha = \{\alpha_j\}$ *be a sequence constructed by Mitjagin's method from any* Φ_p*-binormalizing sequence* $\pi = \{\pi_j\}$. *Then*

$$\lim_{n\to\infty} \Phi_p[\sigma_n(\pi\alpha)] = \infty.$$

PROOF. By (3.15) and (3.8)

$$
\begin{aligned}
n^{-1/p}\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)] &= n^{-1/p}\Phi_p[\sigma_n(\pi)]\cdot Q_n'^{1/p} \\
&\leqq n^{-1/p}\cdot\Phi_p[\sigma_n(\pi)]\cdot Q_n^{1/p} \\
&= n^{-1/p}\cdot\Phi_p[\sigma_n(\pi)]\cdot\frac{n^{1/p}}{\Phi_p[\sigma_n(\pi)]} \\
&= 1,
\end{aligned}
$$

while at the same time, by (3.13), for an infinite number of values $n = n_k$

$$n_k^{-1/p}\cdot\Phi_p[\sigma_{n_k}(\pi)]\cdot\Phi_p[\sigma_{n_k}(\alpha)] = 1.$$

Hence

$$\overline{\lim_{n\to\infty}}\ \{n^{-1/p}\cdot\Phi_p[\sigma_n(\pi)]\cdot\Phi_p[\sigma_n(\alpha)] = 1.$$

So by virtue of Lemma 3.5

$$\lim_{n\to\infty}\Phi_p[\sigma_n(\pi\alpha)] = \Phi_p(\pi\alpha) = \infty.$$

Therefore the lemma is proved.

Now by Lemmas 3.2, 3.6 and Remark 3.3 we have

COROLLARY 3.7. *Let* $\alpha=\{\alpha_j\}$ *be a sequence constructed by Mitjagin's method from any* Φ_p*-binormalizing sequence* $\pi=\{\pi_j\}$. *Then*

$$\mathfrak{S}(p\,;\Pi)^o \subset \mathfrak{S}(p\,;\Pi,\alpha) \subset \mathfrak{S}(p\,;\Pi)$$

and

$$\mathfrak{S}(p\,;\Pi,\alpha) \neq \mathfrak{S}(p\,;\Pi).$$

LEMMA 3.8. *If* $\alpha=\{\alpha_j\}$ *is the same one in Corollary* 3.7, *then*

$$\mathfrak{S}(p\,;\Pi)^o \neq \mathfrak{S}(p\,;\Pi,\alpha).$$

PROOF. By Lemma 3.6 we can choose a sequence $\{M_k\}$ $(M_1 = 0,\ M_k\uparrow\infty)$ of positive integers so that

$$b_k = \sum_{j=M_k+1}^{M_{k+1}}(\pi_j\alpha_j)^p \longrightarrow \infty \qquad (k\longrightarrow\infty). \tag{3.16}$$

We put

$$m_k = M_{k+1} - M_k \qquad (k = 1,2,\cdots) \tag{3.17}$$

and

$$\xi_n = \left\{\frac{1}{m_k}\sum_{j=M_k+1}^{M_{k+1}}\pi_j^p\right\}^{1/p} \qquad (M_k+1\leqq n\leqq M_{k+1}). \tag{3.18}$$

Since $\pi=\{\pi_j\}$, namely $\pi^p=\{\pi_j^p\}$ is non-increasing,

$$\sum_{j=1}^{n} \xi_j^p \leqq \sum_{j=1}^{n} \pi_j^p \qquad (n = 1,2,\cdots),$$

and particularly

$$\sum_{j=1}^{M_k} \xi_j^p = \sum_{j=1}^{M_k} \pi_j^p \qquad (k = 2,3,\cdots).$$

Then we have

$$(3.19) \qquad \lim_{n\to\infty} \{\Phi_p[\sigma_n(\xi)]/\Phi_p[\sigma_n(\pi)]\} = 1 .$$

On the other hand, by the relation (3.15), for all k we have

$$a_k = \sum_{j=M_k+1}^{M_{k+1}} (\xi_j\alpha_j)^p = \frac{1}{m_k} \sum_{j=M_k+1}^{M_{k+1}} \pi_j^p \cdot \sum_{j=M_k+1}^{M_{k+1}} \alpha_j^p$$

$$\leqq \frac{1}{m_k} \sum_{j=1}^{m_k} \pi_j^p \cdot \sum_{j=1}^{m_k} \alpha_j^p$$

$$= \frac{1}{m_k} \sum_{j=1}^{m_k} \pi_j^p \cdot Q'_{m_k}$$

$$\leqq \frac{1}{m_k} \sum_{j=1}^{m_k} \pi_j^p \cdot Q_{m_k} .$$

Hence by the relation (3.8)

$$(3.20) \qquad a_k = \sum_{j=M_k+1}^{M_{k+1}} (\xi_j\alpha_j)^p \leqq 1 \qquad (k = 1,2,\cdots).$$

Thus by (3.16) and (3.20) we have

$$\frac{a_k}{b_k} \longrightarrow 0 \qquad (k \longrightarrow \infty)$$

and

$$\frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} b_k} = \frac{\sum_{j=1}^{M_{n+1}} (\xi_j\alpha_j)^p}{\sum_{j=1}^{M_{n+1}} (\pi_j\alpha_j)^p} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

This means that

$$(3.21) \qquad \frac{\sum_{j=1}^{n} (\xi_j\alpha_j)^p}{\sum_{j=1}^{n} (\pi_j\alpha_j)^p} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

Let X_ξ be any compact operator which has the sequence $\xi = \{\xi_j\}$ given by (3.18) as its s-numbers. Then by the relations (2.12) and (3.19) we have

$$F(X_\xi) = 1,$$

and hence by virtue of Lemma 2.5

$$X_\xi \notin \mathfrak{S}(p\,;\,\Pi)^o.$$

But by the relations (3.1) and (3.21) we have

$$G(X_\xi) = 0$$

and hence by virtue of Lemma 3.2

$$X_\xi \in \mathfrak{S}(p\,;\,\Pi,\alpha).$$

The lemma is therefore proved.

Summarizing Corollary 3.7 and Lemma 3.8, we can state as follows:

THEOREM 3.9. *Let* $1 \leqq p < \infty$ *and let* $\pi = \{\pi_j\}$ *be any* Φ_p*-binormalizing sequence. If* $\alpha = \{\alpha_j\}$ *is the sequence constructed by Mitjagin's method from* $\pi = \{\pi_j\}$, *then three norm ideals* $\mathfrak{S}(p\,;\,\Pi)^o$, $\mathfrak{S}(p\,;\,\Pi,\alpha)$ *and* $\mathfrak{S}(p\,;\,\Pi)$ *with the same symmetric norm* $|X|_{p;\,\Pi}$ *satisfy the following relations:*

$$\mathfrak{S}(p\,;\,\Pi)^o \subset \mathfrak{S}(p\,;\,\Pi,\alpha) \subset \mathfrak{S}(p\,;\,\Pi),$$

$$\mathfrak{S}(p\,;\,\Pi)^o \neq \mathfrak{S}(p\,;\,\Pi,\alpha) \quad \textit{and} \quad \mathfrak{S}(p\,;\,\Pi,\alpha) \neq \mathfrak{S}(p\,;\,\Pi).$$

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