On the Cohomology of the May Complex IV

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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Let X be the May complex derived from the May spectral sequence for $Ext_{P_*}^{*,*}(Z_2, Q_*)$ which is the E_2-term of the algebraic Novikov spectral sequence. This is a differential algebra $Z_2[R_j^i, S_k \mid i \geqq 0, j \geqq 1, k \geqq 0]$ with differential d given by

a). d is a derivation,

b). $d(R_j^i) = \sum_{1 \leq a \leq j-1} R_a^i R_{j-a}^{i+a}$ and $d(S_k) = \sum_{0 \leq a \leq k-1} S_a R_{k-a}^a$.

The original one is the differential subalgebra $Y = Z_2[R_j^i \mid i \geqq 0, j \geqq 1]$ of X. The cohomology of Y, $H^*(Y)$, is partially calculated in [4 : Theorem II. 5. 18] and $H^*(X)$ is partially calculated in [5 : Theorem 18].

The purpose of this paper is to extend the calculations of the defining relations given in [4 : Theorem II. 5. 18] and in [5 : Theorem 18] and to give more simple proofs of [6: Theorem 2.1, 2.2, 3.1, 4.1, 4.2, 5.1] and [7 : Theorem 2, 3, 4, 5]. This is done to define *bracket* (S, N) which is a polynomial in X and to evaluate d *bracket* (S, N). The bracket defined in this paper and the one defined in [7 : Definition 1] differ but both are generalizations of bracket defined in [6 : Definition 1.1].

In section 1, we shall give the definition of *bracket* (S, N) and discuss the properties of this bracket. In section 2, we shall determine the differential of *bracket* (S, N) and in section 3, we shall apply these results to our defining relations. We used a 16-bit small computer HITACHI MB 16001 to conjugate and check the formulas. The language is BASIC. We also have a C program for a mini-computer Toshiba UX 700. The main results are Theorem 2.1 and Theorem 3.1, 3.2, 3.3, ⋯, 3.25.

1. Brackets

In this section we shall define *bracket* (S, N) which is some polynomial in May complex X and discuss its properties.

We first give some notational explanations. Let $n_1, n_2, n_3, \cdots, n_k$ be a finite sequence of non-negative integers. We call this list in this paper and denote it as $[n_1, n_2, n_3, \cdots, n_k]$. We also use the expressions in $[n_1 \mid [n_2, n_3, \cdots, n_k]]$, $[n_1, n_2 \mid [n_3, \cdots, n_k]]$, $\cdots$, $[n_1, n_2, n_3, \cdots, n_k \mid [\,]]$ to show this list. The empty list is denoted as $[\,]$. These notations are commonly used in Prolog [2].

Let L and S be lists and x and y be integers.

The meaning of the expression $x \in L$ is defined as follow:

$x \in [x \mid L]$,
if $x \neq y$ then $x \in [y \mid L]$ if and only if $x \in L$.

In this case we call x an item of L. We define *rest* (L, x) and *rest* (L, S) to be the list given by removing, respectively, all x and all items contained in S from L:

$rest\ ([\,], x) = [\,]$,
$rest\ ([x \mid L], x) = rest\ (L, x)$,
$rest\ ([y \mid L], x) = [y \mid rest\ (L, x)]$ if $x \neq y$,
$rest\ (L, [\,]) = L$,
$rest\ (L, [x \mid S]) = rest\ (rest\ (L, x), S)$,

and *del* (L, x) and *del* (L, S) to be the list given by removing, respectively, one x and each item contained in S from L:

$del\ ([\,], x) = [\,]$,
$del\ ([x \mid L], x) = L$,
$del\ ([y \mid L], x) = [y \mid del\ (L, x)]$ if $x \neq y$,
$del\ (L, [\,]) = L$,
$del\ (L, [x \mid S]) = del\ (del\ (L, x), S)$,

and *set* (L) to be the list which consists by removing the overlapped items from L:

$set\ ([\,]) = [\,]$,
$set\ ([x \mid S]) = [x \mid set\ (rest\ (S, x))]$,

and $\# L$ to be the length of L:

$\# [\,] = 0$,
$\# [x \mid L] = \# L + 1$,

and *ind* (L, x) to be the number of x contained in L:

$ind\ ([\,], x) = 0$,
$ind\ ([x \mid L], x) = ind\ (L, x) + 1$,
$ind\ ([y \mid L], x) = ind\ (L, x)$ if $x \neq y$,

and *item* (L, x), $1 \leqq x \leqq \# L$, to be the x-th item of L:

$item\ ([y \mid L], 1) = y$,
$item\ ([y \mid L], x) = item\ (L, x-1)$ *if* $x > 1$,

and *itel* (x, y) to be the list with length y consisted by x only:

$itel\ (x, 0) = [\,]$
$itel\ (x, y) = [x \mid itel\ (x, y-1)]$ if $y > 0$,

and $L \cup S$ to be the list L followed by S:

$[\,] \cup S = S$,

$[x \mid L] \cup S = [x \mid L \cup S]$,

and $L \cap S$ to be the list consisted by items contained in both lists:

$[\,] \cap S = [\,]$,
$[x \mid L] \cap S = [x \mid L \cap del\ (S, x)]$ if $x \in S$,
$[x \mid L] \cap S = L \cap S$ if $x \notin S$.

The meaning of the expression $L \subset S$ is defined as follow:

$[\,] \subset S$,
$[x \mid L] \subset S$ if and only if $x \in S$ and $L \subset del\ (S, x)$.

In this case we call L a sublist of S.

For the simplicity, we let R^i_j be 0 if j is less than or equal to 0 . We generalize S_k and R^i_j to S_N, R^i_{A-i} and R^A_{n-A} , respectively. These are defined inductively as follow:

$$S_{[\,]} = 1\,,$$

$$S_{[n \mid N]} = S_n\, S_N\,,$$

$$R^i_{[\,]-i} = 1\,,$$

$$R^i_{[a \mid A]-i} = R^i_{a-i}\ R^i_{A-i}\,,$$

$$R^{[\,]}_{n-[\,]} = 1\,,$$

$$R^{[a \mid A]}_{n-[a \mid A]} = R^a_{n-a}\ R^A_{n-A}\ .$$

The brackets in [5 : Definition 1.1.] are generalized in the following form.

DEFINITION 1.1. For any two lists S and N of non-negative integers, we define *bracket* (S, N) in X inductively as follow:

$$bracket\ (S, N) = 0 \text{ if } \# S > \# N,$$

$$bracket\ ([\,], N) = S_N,$$

$$bracket\ ([s \mid T], N) = \sum_{\substack{A \subset N \\ \# A = ind\,(S,\,s)}} R^s_{A-s}\ bracket\ (rest\ (S, s),\ del\ (N, A)) \text{ if } S = [s \mid T].$$

In the above summation, A runs through all sublists, which differ under the permutations of A, of N with $\# A = ind\ (S, s)$. Since $R^s_{A'-s} = R^s_{A-s}$ and $del\ (N, A') = del\ (N, A)$ for each permutated list A' of A, our definition is well-defined. We also define $bracket_{R^i_{n-i}:even}\ (S, N)$, $bracket_{S_n:even}\ (S, N)$, $bracket_{R^i_{n-i}:odd}\ (S, N)$ and $bracket_{S_n:odd}\ (S, N)$ to be the summations of the nomomials contained in *bracket* (S, N) which contain even R^i_{n-i}, even S_n, odd R^i_{n-i} and odd S_n, respectively.

REMARK. By the above definition, we have that $S_k = bracket\ ([\,], [k])$, $R^i_j = bracket\ ([i], [i+j])$, May's indecomposable element b_{ij} is represented by bracket $([i, i], [i+j, i+j])$ for

$i \geqq 0$ and $j \geqq 2$ and $h_i(n_1, n_2, \cdots, n_k)$ is represented by *bracket* (S, N), where $S = [i, i+n_1, i+n_2, \cdots, i+n_k]$ ($S=[i]$ if $k=0$) and $N = del([i, i+1, i+2, \cdots, i+2k+1], S)$, for $i \geqq 0$, $k \geqq 0$, $n_1 = 1$ and $n_{j-1} < n_j \leqq 2j-1$ $(2 \leqq j \leqq k)$. Similarly, another indecomposable element a_k is represented by *bracket* $([\,], [k, k])$ for $k \geqq 1$ and $g(n_0, n_1, \cdots, n_k)$ is represented by *bracket* (S, N), where $S = [n_0, n_1, \cdots, n_k]$ ($S = [\,]$ if $k = -1$) and $N = del([0, 1, 2, \cdots, 2k+2], S)$, for $k \geqq -1$, $n_0 = 0$ and $n_{j-1} < n_j \leqq 2j$ $(1 \leqq j \leqq k)$.

LEMMA 1.1. *Let S and N be lists of non-negative integers and let S′ and N′ be the permutated lists of S and N, respectively. Then we have bracket* $(S', N') =$ *bracket* (S, N).

PROOF. We first prove *bracket* $(S, N') =$ *bracket* (S, N) by the induction on $\# N$. If $\# N = 0$ then the result is clear. So we assume $\# N > 0$. Since the symmetry group is generated by transposition $(i, i+1)$, we only prove in the case $N' = (i, i+1)N$. If $S = [\,]$ then the result is clear. So let $S = [s \mid T]$ and $N = [n_1, n_2, \cdots, n_k]$ with $n_i \neq n_{i+1}$. By the definition of bracket, we have

$$bracket\,(S, N) = \sum_{\substack{A \subset N \\ \# A = ind(S, s)}} R^{s}_{A-s}\, bracket\,(rest\,(S, s), del\,(N, A))$$

and

$$bracket\,(S, N') = \sum_{\substack{B \subset N' \\ \# B = ind(S, s)}} R^{s}_{B-s}\, bracket\,(rest\,(S, s), del\,(N', B)).$$

Here, the set of sublists A and set of sublists B are same up to permutation and if $n_i \in A$ or $n_{i+1} \in A$ then $del(N, A) = del(N', A)$ and if $n_i \notin A$ and $n_{i+1} \notin A$ then *bracket* $(rest(S, s), del(N, A)) =$ *bracket* $(rest(S, s), del(N', A))$ by the hypothesis of induction. Therefore we have *bracket* $(S, N) =$ *bracket* (S, N').

Next we prove *bracket* $(S', N) =$ *bracket* (S, N). This is also sufficient to show that the transpositions $(i, i+1)$ on S does not change brackets. We prove this by the induction on i.

If $\# S < 2$ then the result is clear. So we assume that $\# S \geqq 2$. Let $S = [s_1, s_2 \mid T]$ and $S' = (1, 2)S$. If $s_1 = s_2$ then the result is clear. So we assume that $s_1 \neq s_2$. By the definition of bracket and the fact that $rest(rest(S, s_1), s_2) = rest(S, [s_1, s_2])$ and $del(del(N, A), B) = del(N, A \cup B)$, we have that

$$bracket\,(S, N)$$

$$= \sum_{\substack{A \subset N \\ \# A = ind(S, s_1)}} R^{s_1}_{A-s_1}\, bracket\,(rest\,(S, s), del\,(N, A))$$

$$= \sum_{\substack{A \subset N \\ \# A = ind(S, s_1)}} R^{s_1}_{A-s_1} \sum_{\substack{B \subset del(N, A) \\ \# B = ind(rest(S, s_1), s_2)}} R^{s_2}_{B-s_2}\, bracket\,(rest\,(S, [s_1, s_2]), del\,(N, A \cup B)).$$

Since $ind(rest(S, s_1), s_2) = ind(S', s_2)$, $ind(S, s_1) = ind(rest(S', s_2), s_1)$ and *bracket* $(rest(S', [s_1, s_2]), del(N, A \cup B)) =$ *bracket* $(rest(S', [s_1, s_2]), del(N, B \cup A))$, we have that

$$= \sum_{\substack{B \subset N \\ \# B = ind(S', s_2)}} R^{s_2}_{B-s_2} \sum_{\substack{A \subset del(N, B) \\ \# A = ind(rest(S', s_2), s_1)}} R^{s_1}_{A-s_1}\ bracket\ (rest\ (S', [s_1, s_2]), del\ (N, B \cup A))$$

$$= \sum_{\substack{B \subset N \\ \# B = ind(S', s_2)}} R^{s_2}_{B-s_2}\ bracket\ (rest\ (S', s_2), del\ (N, B))$$

$$= bracket\ (S', N).$$

Then the transposition $(1, 2)$ of S does not change the brackets. Next we assume that if $i < j$ ($j \geq 2$) then the transposition $(i, i+1)$ of S does not change the brackets. Let $S' = (j, j+1)S$ and $S = [s \mid T]$ and $S' = [s \mid T']$. Then we have $T' = (j-1, j)T$. By the definition of brackets, we have that

$$bracket\ (S, N) = \sum_{\substack{A \subset N \\ \# A = ind(S, s)}} R^{s}_{A-s}\ bracket\ (rest\ (T, s), del\ (N, A))$$

and

$$bracket\ (S', N) = \sum_{\substack{A \subset N \\ \# A = ind(S', s)}} R^{s}_{A-s}\ bracket\ (rest\ (T', s), del\ (N, A)).$$

Here we have that $ind\ (S, s) = ind\ (S', s)$. Let $t_{j-1} = item\ (T, j-1)$ and $t_j = item\ (T, j)$. If $s = t_{j-1}$ or $s = t_j$ then $rest\ (T, s) = rest\ (T', s)$ and then $bracket\ (rest\ (T, s), del\ (N, A)) = bracket\ (rest\ (T', s), del\ (N, A))$ else there exist an integer k with $k < j$ such that $rest\ (T', s) = (k-1, k)\ rest\ (T, s)$ and thereforeby the hypothesis of induction, we have that $bracket\ (rest\ (T, s), del\ (N, A)) = bracket\ (rest\ (T', s), del\ (N, A))$. Then we have the result.

LEMMA 1.2. *Let S and N be lists of non-negative integers with $\# N \geq \# S$. Then we have the following expansion formulas.*

i). *bracket* (S, N)

$$= \sum_{\substack{A \subset N \\ \# A = ind(S, s)}} R^{s}_{A-s}\ bracket\ (rest\ (S, s), del\ (N, A))$$

for each item s of S.

ii). *bracket* (S, N)

$$= \sum_{\substack{A \subset N \\ \# A = \# N - \# S}} S_A\ bracket\ (S, del\ (N, A)).$$

iii). *bracket* (S, N)

$$= \sum_{\substack{A \subset S \\ \# A \leq ind(N, n)}} R^{A}_{n-A}\ S_n^{ind(N, n) - \# A}\ bracket\ (del\ (S, A), rest\ (N, n))$$

for each item n of N.

PROOF. i). This is easily induced from Lemma 1.1.

ii). This is proved by the induction on $\# N$. If $\# N = 0$ or $S = [\,]$ then the result is clear. So we assume that $\# N > 0$ and $S = [s \mid T]$. By the definition of bracket, we have that

$$\mathit{bracket}\ ([s \mid T],\ N)$$

$$= \sum_{\substack{A \subset N \\ \# A = \mathit{ind}(S,s)}} R^{s}_{A-s}\ \mathit{bracket}\ (\mathit{rest}\ (S,\ s),\ \mathit{del}\ (N,\ A)).$$

Since $\#\mathit{del}\ (N,\ A) < \# N$, we have, by the hypothesis of induction, that

$$= \sum_{\substack{A \subset N \\ \# A = \mathit{ind}(S,s)}} R^{s}_{A-s} \sum_{\substack{B \subset \mathit{del}(N,A) \\ \# B = \#\mathit{del}(N,A) - \#\mathit{rest}(S,s)}} S_B\ \mathit{bracket}\ (\mathit{rest}\ (S,\ s),\ \mathit{del}\ (\mathit{del}\ (N,\ A),\ B)).$$

Since $\#\mathit{del}\ (N,\ A) = \# N - \# A$ and $\#\mathit{rest}\ (S,\ s) = \# S - \mathit{ind}\ (S,\ s)$, we have that

$$= \sum_{\substack{A \cup B \subset N \\ \# A = \mathit{ind}(S,s) \\ \# B = \# N - \# S}} R^{s}_{A-s} S_B\ \mathit{bracket}\ (\mathit{rest}\ (S,\ s),\ \mathit{del}\ (N,\ A \cup B))$$

$$= \sum_{\substack{B \subset N \\ \# B = \# N - \# S}} S_B \sum_{\substack{A \subset \mathit{del}(N,B) \\ \# A = \mathit{ind}(S,s)}} R^{s}_{A-s}\ \mathit{bracket}\ (\mathit{rest}\ (S,\ s),\ \mathit{del}\ (\mathit{del}\ (N,\ B),\ A))$$

$$= \sum_{\substack{B \subset N \\ \# B = \# N - \# S}} S_B\ \mathit{bracket}\ (S,\ \mathit{del}\ (N,\ B)).$$

iii). We shall prove the induction on $\# S$. Let $S = [\,]$. Then we have that

$$\mathit{bracket}\ ([\,],\ N) = S_N = S_n^{\mathit{ind}(N,n)} S_{\mathit{rest}(N,n)} = S_n^{\mathit{ind}(N,n)}\ \mathit{bracket}\ ([\,],\ \mathit{rest}\ (N,\ n)).$$

Then the formula is valid in the case $S = [\,]$. From now on we also use abbreviations *b*( ), *r*( ), *i*( ), *il*( ) and *d*( ) for *bracket*( ), *ind*( ), *itel*( ) and *del*( ), respectively. Now let $S = [s \mid T]$. By the definition of bracket, we have that

$$\mathit{bracket}\ (S,\ N)$$

$$= \sum_{\substack{B \subset N \\ \# B = \mathit{ind}(S,s)}} R^{s}_{B-s}\ \mathit{bracket}\ (\mathit{rest}\ (S,\ s),\ \mathit{del}\ (N,\ B))$$

$$= \sum_{i \geq 0} \sum_{\substack{B \subset N \\ \# B = \mathit{ind}(S,s) \\ \mathit{ind}(B,n) = i}} R^{s}_{B-s}\ \mathit{bracket}\ (\mathit{rest}\ (S,\ s),\ \mathit{del}\ (N,\ B)).$$

By the hypothesis of induction, we have that

$$= \sum_{i \geq 0} \sum_{\substack{B \subset N \\ \# B = i(S,s) \\ i(B,n) = i}} R^{s}_{B-s} \sum_{\substack{A \subset r(S,s) \\ \# A \leq i(d(N,B),n)}} R^{A}_{n-A}\ S_n^{i(d(N,B),n) - \# A}\ \ b\ (d\ (r\ (S,\ s),\ A),\ r\ (d\ (N,\ B),\ n)).$$

If $\mathit{ind}\ (B,\ n) = i$ then $\mathit{ind}\ (\mathit{del}\ (N,\ B),\ n) = \mathit{ind}\ (N,\ n) - \mathit{ind}\ (B,\ n) = \mathit{ind}\ (N,\ n) - i$. Then we have that

$$= \sum_{i \geq 0} \sum_{\substack{B \subset N \\ \# B = i(S,s) \\ i(B,n) = i}} \sum_{\substack{A \subset r(S,s) \\ \# A \leq i(N,n) - i}} R^{s}_{B-s}\ R^{A}_{n-A}\ S_n^{i(N,n) - i - \# A}\ \ b\ (d\ (r\ (S,\ s),\ A),\ r\ (d\ (N,\ B),\ n))$$

$$= \sum_{i \geq 0} \sum_{\substack{A \subset r(S,s) \\ \# A \leq i(N,n) - i}} R^{A}_{n-A}\ S_n^{i(N,n) - i - \# A} \sum_{\substack{B \subset N \\ \# B = i(S,s) \\ i(B,n) = i}} R^{s}_{B-s}\ \ b\ (d\ (r\ (S,\ s),\ A),\ r\ (d\ (N,\ B),\ n))$$

$$= \sum_{i \geq 0} \sum_{\substack{A \subset r(S,s) \\ \# A \leq i(N,n) - i}} R^{A}_{n-A}\ S_n^{i(N,n) - i - \# A} \sum_{\substack{B \subset N \\ \# B = i(S,s) \\ i(B,n) = i}} (R^{s}_{n-s})^{i}\ R^{s}_{r(B,n)-s}\ \ b\ (d(r\ (S,s),\ A), r\ (d(N,B),n))$$

$$=\sum_{i\geq 0}\ \sum_{\substack{A\subset r(S,s)\\ \sharp A\leq i(N,n)-i}} R^{A\ il(s,i)}_{n-A\ il(S,i)}\ S_n^{i(N,n)-i-\sharp A} \sum_{\substack{B\subset N\\ \sharp B=i(S,s)\\ i(B,n)=i}} R^{s}_{r(B,n)-s}\cdot b\,(d\,(r\,(S,s),A),\ r\,(d\,(N,B),n))$$

Since *ind* $(S, s)-i=$ *ind* $(S, s)-$*ind* $(A, s)=$ *ind* (*del* (S, A), s), *del* (*rest* (S, s), A) $=$ *rest* (*del* (S, A), s) and *rest* (*del* (N, B), n) $=$ *del* (*rest* (N, n), B), we have that

$$=\sum_{i\geq 0}\ \sum_{\substack{A\subset S\\ \sharp A\leq i(N,n)\\ i(A,s)=i}} R^{A}_{n-A}\ S_n^{i(N,n)-\sharp A} \sum_{\substack{B\subset r(N,n)\\ \sharp B=i(d(N,A),s)}} R^{s}_{B-s}\ b\,(r\,(d\,(S,A),s),\ d\,(r\,(N,n),B))$$

$$=\sum_{i\geq 0}\ \sum_{\substack{A\subset S\\ \sharp A\leq ind(N,n)\\ ind(A,s)=i}} R^{A}_{n-A}\ S_n^{ind(N,n)-\sharp A}\ bracket\,(del\,(S,A),\ rest\,(N,n))$$

$$=\sum_{\substack{A\subset S\\ \sharp A\leq ind(N,n)}} R^{A}_{n-A}\ S_n^{ind(N,n)-\sharp A}\ bracket\,(del\,(S,A),\ rest\,(N,n)).$$

Then we have the result.

LEMMA **1.3.** *We have the following formula.*

i). *If* $i\neq k$ *then we have that*

$$\sum_{n\in set(N)} R^{k}_{n-k}\ bracket_{R^i_{n-i}:even}\,(S,\ del\,(N,n))$$

$$=\sum_{\substack{A\cup B\subset N\\ \sharp A=ind(S,i)\\ \sharp B=ind(S,k)+1\\ \sharp set(B)-\sharp\{n\in set(B)\cap set(A)\mid ind(A,n):odd\}:odd}} R^{i}_{A-i}\ R^{k}_{B-k}\ bracket\,(rest\,(S,[i,k]),\ del\,(N,A\cup B))$$

ii). $$\sum_{n\in set(N)} R^{k}_{n-k}\ bracket_{S_n:even}\,(S,\ del\,(N,n))$$

$$=\sum_{\substack{A\cup B\subset N\\ \sharp A=\sharp N-\sharp S-1\\ \sharp B=ind(S,k)+1\\ \sharp set(B)-\sharp\{n\in set(B)\cap set(A)\mid ind(A,n):odd\}:odd}} S_A\ R^{k}_{B-k}\ bracket\,(rest\,(S,k),\ del\,(N,A\cup B)).$$

iii). *If* $n\neq m$ *then we have that*

$$\sum_{i\in set(S)} R^{i}_{n-i}\ bracket_{R^i_{m-i}:even}\,(del\,(S,i),\ N)$$

$$=\sum_{\substack{A\cup B\subset N\\ 1\leq\sharp A\leq ind(N,n)+1\\ \sharp B\leq ind(N,m)\\ \sharp set(A)-\sharp\{i\in set(B)\cap set(A)\mid ind(B,i):odd\}:odd}} R^{A}_{n-A}\ R^{B}_{m-B}\ S_n^{ind(N,n)-\sharp A+1}\ S_m^{ind(N,m)-\sharp B}\ brucket\,(del\,(S,A\cup B),$$

$$rest\,(N,[n,m])).$$

iv). *If* $n\neq m$ *then we have that*

$$S_n\ bracket_{S_m:even}\,(S,N)$$

$$=\sum_{\substack{A\cup B\subset S\\ \sharp A\leq ind(N,n)\\ \sharp B\leq ind(N,m)\\ ind(N,m)-\sharp B:even}} R^{A}_{n-A}\ R^{B}_{m-B}\ S_n^{ind(N,n)-\sharp A+1}\ S_m^{ind(N,m)-\sharp B}\ bracket\,(del\,(S,A\cup B),\ rest\,(N,[n,m])).$$

v). *If* $n \neq m$ *then we have that*

$$\sum_{i \in set(S)} R^{i}_{n-i}\ bracket_{R^{i}_{m-i}:even}(del(S, i), N) + S_n\ bracket_{S_m:even}(S, N)$$

$$= \sum_{\substack{A \cup B \subset S \\ \# A = ind(N,n)+1 \\ \# B \leq ind(N,m) \\ \# set(A) - \# \{i \in set(A) \cap set(B) \mid ind(B,i) : odd\} : odd}} R^{A}_{n-A}\ R^{B}_{m-B}\ S_m^{ind(N,m)-\# B}\ bracket(del(S, A \cup B),$$

$$rest(N, [n, m]))$$

$$+ \sum_{\substack{A \cup B \subset S \\ 1 \leq \# A \leq ind(N,n) \\ \# B \leq ind(N,m) \\ \# set(A) - \# \{i \in set(A) \cap set(B) \mid ind(B,i) : odd\} : odd \text{ and } ind(N,m) - \# B : odd \text{ or} \\ \# set(A) - \# \{i \in set(A) \cap set(B) \mid ind(B,i) : odd\} : even \text{ and } ind(N,m) - \# B : even}} R^{A}_{n-A}\ R^{B}_{m-B}\ S_n^{ind(N,n)-\# A+1}\ S_m^{ind(N,m)-\# B}$$

$$bracket(del(S, A \cup B), rest(N, [n, m]))$$

$$+ \sum_{\substack{B \subset S \\ \# B < ind(N,m) \\ ind(N,m) - \# B : even}} R^{B}_{m-B}\ S_n^{ind(N,n)+1}\ S_m^{ind(N,m)-\# B}\ bracket(del(S, B), rest(N, [n, m])).$$

PROOF. i). By the definition of bracket and Lemma 1.2.i), we have that

$$\sum_{n \in set(N)} R^{k}_{n-k}\ bracket_{R^{i}_{n-i}:even}(S, del(N, n))$$

$$= \sum_{n \in set(N)} R^{k}_{n-k} \sum_{\substack{A \subset del(N,n) \\ \# A = ind(S,i) \\ ind(A,n) : even}} R^{i}_{A-i}\ bracket(rest(S, i), del(N, [n \mid A]))$$

$$= \sum_{n \in set(N)} \sum_{\substack{A \subset d(N,n) \\ \# A = i(S,i) \\ i(A,n) : even}} \sum_{\substack{B \subset d(N,[n \mid A]) \\ \# B = i(r(S,i),k)}} R^{i}_{A-i}\ R^{k}_{[n \mid B]-k}\ b(r(S, [i, k]), d(N, [n \mid A \cup B]))$$

$$= \sum_{n \in set(N)} \sum_{\substack{B \subset d(N,n) \\ \# B = i(S,k)}} \sum_{\substack{A \subset d(N,[n \mid B]) \\ \# A = i(S,i) \\ i(A,n) : even}} R^{i}_{A-i}\ R^{k}_{[n \mid B]-k}\ b(r(S, [i, k]), d(N, [n \mid A \cup B])).$$

Now we decompose $[n \mid B]$ and A into $B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_{\lambda+\mu+\nu}$ and $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{\mu+\nu+\rho}$, respectively, satisfying that each B_i and A_i is consisted by a single item and $B_i \cap B_j =$ [] and $A_i \cap A_j =$ [] for $i \neq j$, $B_i \cap A =$ [] for $1 \leq i \leq \lambda$, $B_{\lambda+i} \cap A_i \neq$ [] *for* $1 \leq i \leq \mu + \nu$, $B \cap A_i =$ [] for $\mu + \nu + 1 \leq i \leq \mu + \nu + \rho$, $\# A_i$ is even for $1 \leq i \leq \mu$ and $\# A_i$ is odd for $\mu + 1 \leq i \leq \mu + \nu$. Since *ind* (A, n) is even, n must be contained in B_i for some i with $1 \leq i \leq \lambda + \mu$. If $\lambda + \mu$ is even then the summation of monomials contain R^{k}_{n-k} is zero else it is non zero. Since the condition $\lambda + \mu$ is odd is equivalent to the condition $\# set(B) - \# \{n \in set(A) \cap set(B) \mid ind(A, n) : odd\}$ is odd, we have that

$$= \sum_{\substack{A \cup B \subset N \\ \# A = ind(S,i) \\ \# B = ind(S,k)+1 \\ \# set(B) - \# \{n \in set(A) \cap set(B) \mid ind(A,n) : odd\} : odd}} R^{i}_{A-i}\ R^{k}_{B-k}\ bracket(rest(S, [i,k]), del(N, A \cup B)).$$

ii). This is proved by the essentially same methods. By the definition of bracket and Lemma 1.2.ii), we have that

$$\sum_{n \in set(N)} R^{k}_{n-k}\ bracket_{S_n:even}(S, del(N, n))$$

$$= \sum_{n\in set(N)} \sum_{\substack{B\subset del(N,n) \\ \# B=ind(S,k)}} R^{k}_{[n\mid B]-k} \; bracket_{S_n:even} \, (rest\,(S, k), del\,(N, [n \mid B]))$$

$$= \sum_{n\in set(N)} \sum_{\substack{B\subset d(N,n) \\ \# B=i(S,k)}} \sum_{\substack{A\subset d(N,[n\mid B]) \\ \# A=\# N-\# S-1 \\ i(A,n):even}} R^{k}_{[n\mid B]-k} \; S_A \; b\,(r\,(S, k), d\,(N, [n \mid A\cup B]))$$

Now we decompose $[n \mid B]$ and A into $B_1\cup B_2\cup B_3\cup\cdots\cup B_{\lambda+\mu+\nu}$ and $A_1\cup A_2\cup A_3\cup\cdots\cup A_{\mu+\nu+\rho}$, respectively, satisfying that each B_i and A_i is consisted by a single item and $B_i\cap B_j = [\,]$ and $A_i\cap A_j = [\,]$ for $i\neq j$, $B_i\cap A = [\,]$ for $1\leqq i\leqq \lambda$, $B_{\lambda+i}\cap A_i\neq [\,]$ for $1\leqq i\leqq \mu+\nu$, $B\cap A_i = [\,]$ for $\mu+\nu+1\leqq i\leqq \mu+\nu+\rho$, $\# A_i$ is even for $1\leqq i\leqq \mu$ and $\# A_i$ is odd for $\mu+1\leqq i\leqq \mu+\nu$. Since $ind\,(A, n)$ is even, n must be contained in B_i for some i with $1\leqq i\leqq \lambda+\mu$. If $\lambda+\mu$ is even then the summation of monomials contain R^k_{n-k} is zero else it is non zero. Since the condition $\lambda+\mu$ is odd is equivalent to the condition $\# set\,(B) - \# \{n\in set\,(A)\cap set\,(B) \mid ind\,(A, n) : odd\}$ is odd, we have that

$$= \sum_{\substack{A\cup B\subset N \\ \# A=\# N-\# S-1 \\ \# B=ind(S,k)+1 \\ \# set(B)-\#\{n\in set(A)\cap set(B) \mid ind(A,n):odd\}:odd}} R^{k}_{[n\mid B]-k} \; S_A \; bracket\,(rest\,(S, k), del\,(N, A\cup B)).$$

iii). This is also proved by the essentially same methods. By the definition of bracket and Lemma 1.2.iii), we have that

$$\sum_{i\in set(S)} R^i_{n-i} \; bracket_{R^i_{m-i}:even} \, (del\,(S, i), N)$$

$$= \sum_{i\in set(S)} \sum_{\substack{A\subset del(S,i) \\ \# A\leqq ind(N,n)}} R^{[i\mid A]}_{n-[i\mid A]} \; S_n^{ind(N,n)-\# A} \; bracket\,(del\,(S, [i \mid A]), rest\,(N, n))$$

$$= \sum_{i\in set(S)} \sum_{\substack{A\subset d(S,i) \\ \# A\leqq i(N,n)}} \sum_{\substack{B\subset d(S,[i\mid A]) \\ \# B\leqq i(r(N,n),m) \\ i(B,i):even}} R^{[i\mid A]}_{n-[i\mid A]} \; R^B_{m-B} \; S_n^{i(N,n)-\# A}$$

$$\times S_m^{i(N,m)-\# B} \; b\,(d\,(S, [i \mid A\cup B]), r\,(N, [n, m])).$$

Now we decompose $[i \mid A]$ and B into $A_1\cup A_2\cup A_3\cup\cdots\cup A_{\lambda+\mu+\nu}$ and $B_1\cup B_2\cup B_3\cup\cdots\cup B_{\mu+\nu+\rho}$, respectively, satisfying that each A_i and B_i is consisted by a single item and $A_i\cap A_j = [\,]$ and $B_i\cap B_j = [\,]$ for $i\neq j$, $A_i\cap B = [\,]$ for $1\leqq i\leqq \lambda$, $A_{\lambda+i}\cap B_i\neq [\,]$ for $1\leqq i\leqq \mu+\nu$, $A\cap B_i = [\,]$ for $\mu+\nu+1\leqq i\leqq \mu+\nu+\rho$, $\# B_i$ is even for $1\leqq i\leqq \mu$ and $\# B_i$ is odd for $\mu+1\leqq i\leqq \mu+\nu$. Since $ind\,(B, i)$ is even, i must be contained in A_j for some j with $1\leqq j\leqq \lambda+\mu$. If $\lambda+\mu$ is even then the summation of monomials contain R^i_{n-i} is zero else it is non-zero. Since the condition $\lambda+\mu$ is odd is equivalent to the condition $\# set\,(A) - \# \{i\in set\,(A)\cap set\,(B) \mid ind\,(B, i) : odd\}$ is odd, we have that

$$= \sum_{\substack{A\cup B\subset S \\ 1\leqq \# A\leqq ind(N,n)+1 \\ \# B\leqq ind(N,m) \\ \# set(A)-\#\{i\in set(A)\cap set(B) \mid ind(B,i):odd\}:odd}} R^A_{n-A} \, R^B_{m-B} \, S_n^{ind(N,n)-\# A+1} \; S_m^{ind(N,m)-\# B} \; bracket\,(del\,(S, A\cup B),$$

$$rest\,(N, [n, m])).$$

iv). By the definition of bracket and Lemma 1.2.iii), we have that

$$S_n \; bracket_{S_m:even} \, (S, N)$$

$$=\sum_{\substack{A\subset S\\ \# A\leq ind(N,n)}} R^{A}_{n-A}\ S_{n}^{ind(N,n)-\# A+1}\ bracket_{S_m:even}\ (del\,(S,A),\, rest\,(N,n))$$

$$=\sum_{\substack{A\subset S\\ \# A\leq i(N,n)}}\ \sum_{\substack{B\subset d(S,A)\\ \# B\leq i(N,m)\\ i(N,m)-\# B:\, even}} R^{A}_{n-A}\ R^{B}_{m-B}\ S_{n}^{i(N,n)-\# A+1}\ S_{m}^{i(N,m)-\# B}\ b\,(d\,(S, A\cup B),\, r\,(N,[n,m])).$$

Then we have the result.

v). This is easily induced by iii) and iv).

2. Differentials of brackets

In this section we shall determine the differentials of the brackets. From now on we also use abbreviations $b(\ )$, $d(\ )$, $i(\ )$, $r(\ )$ and $s(\ )$ for *bracket*( ), *del*( ), *ind*( ), *rest*( ) and *set*( ), respectively.

THEOREM 2.1. *Let S and N be lists of non-negative integers such that* $\# S\leqq \# N$. *Then we have that*

$$d\ bracket\,(S, N)$$

$$=\sum_{\substack{a\in s(S)\\ a\notin N}}\ \sum_{\substack{B\subset N\\ \# B=i(S,a)+1}} \Big\{\sum_{i\in s(B)}\ \sum_{\substack{A\subset d(N,B)\\ \# A=i(S,i)-1\\ \# s(B)-\#\{n\in s(A)\cap s(B)\mid i(A,n):\,odd\}:\,odd}} R_{\{a\mid A\}-i}\ b\,(r\,(S,[i,\ a]),\, d\,(N, A\cup B))$$

$$+\sum_{\substack{C\subset d(N,B)\\ \# C=\# N-\# S-1\\ \# s(B)-\#\{n\in s(B)\cap s(C)\mid i(C,n):\,odd\}:\,odd}} S_{[a\mid C]}\ b\,(r\,(S,\ a),\, d\,(N, B\cup C))\Big\}$$

$$+\sum_{a\notin N\cup S}\ \sum_{n\in s(N)} R^{a}_{n-a}\ b\,(S,[\,a\mid d\,(N,n)])$$

$$+\sum_{a\notin N\cup S}\ \sum_{n\in s(N)} R^{a}_{n-a}\ \Big\{\sum_{i\in s(S)} R^{i}_{a-i}\ b_{R^{i}_{n-i}:\,odd}\,(d\,(S,i),\, d\,(N,n))$$

$$+S_{a}\ b_{S_n:\,odd}\,(S,\, d\,(N,n))\Big\}$$

$$+\sum_{a\in N}\ \sum_{n\in s(N)} R^{a}_{n-a}\ \Big\{\sum_{\substack{A\cup B\subset S\\ \# A=i(N,a)+1\\ \# B\leq i(N,n)-1\\ \# s(A)-\#\{i\in s(A)\cap s(B)\mid i(B,i):\,odd\}:\,odd}} R^{A}_{a-A}\ R^{B}_{n-B}\ S_{n}^{i(N,n)-\# B-1}$$

$$b\,(d\,(S, A\cup B),\, r\,(N,[n,\ a]))$$

$$+\sum_{\substack{A\cup B\subset S\\ 1\leq \# A\leq i(N,a)\\ \# B\leq i(N,n)-1\\ \# s(A)-\#\{i\in s(A)\cap s(B)\mid i(B,i):\,odd\}:\,odd\ and\ i(N,n)-\# B-1:\,odd\ or\\ \# s(A)-\#\{i\in s(A)\cap s(B)\mid i(B,i):\,odd\}:\,even\ and\ i(N,n)-\# B-1:\,even}} R^{A}_{a-A}\ R^{B}_{n-B}\ S_{a}^{i(N,a)-\# A+1}\ S_{n}^{i(N,n)-\# B-1}$$

$$b\,(d\,(S, A\cup B),\, r\,(N,[n,\ a]))$$

$$+\sum_{\substack{B\subset S\\ \# B\leq i(N,n)-1\\ i(N,n)-\# B-1:\,even}} R^{B}_{n-B}\ S_{a}^{i(N,a)+1}\ S_{n}^{i(N,n)-\# B-1}\ b\,(d\,(S,B),\, r\,(N,[n,\ a]))\Big\}.$$

PROOF. Since X is a commutative differential algebra over Z_2 and the differential d is a derivation, we have that

$d\ bracket\ (S, N)$

$$= \sum_{i \in set(S)} \sum_{n \in set(N)} d\ (R^{i}_{n-i})\ bracket_{R^{i}_{n-i}:even}\ (del\ (S, i), del\ (N, n))$$

$$+ \sum_{n \in set(N)} d\ (S_n)\ bracket_{S_n:even}\ (S, del\ (N, n))$$

$$= \sum_{i \in s(S)} \sum_{n \in s(N)} \sum_{\substack{a \in s(S) \\ a \notin s(N)}} R^{i}_{a-i}\ R^{a}_{n-a}\ b_{R^{i}_{n-i}:even}\ (d\ (S, i), d\ (N, n)) \quad (1)$$

$$+ \sum_{i \in s(S)} \sum_{n \in s(N)} \sum_{\substack{a \notin s(S) \\ a \notin s(N)}} R^{i}_{a-i}\ R^{a}_{n-a}\ b_{R^{i}_{n-i}:even}\ (d\ (S, i), d\ (N, n)) \quad (2)$$

$$+ \sum_{i \in s(S)} \sum_{n \in s(N)} \sum_{a \in s(N)} R^{i}_{a-i}\ R^{a}_{n-a}\ b_{R^{i}_{n-i}:even}\ (d\ (S, i), d\ (N, n)) \quad (3)$$

$$+ \sum_{n \in s(N)} \sum_{\substack{a \in s(S) \\ a \notin s(N)}} S_a\ R^{a}_{n-a}\ b_{S_n:even}\ (S, d\ (N, n)) \quad (4)$$

$$+ \sum_{n \in s(N)} \sum_{\substack{a \notin s(S) \\ a \notin s(N)}} S_a\ R^{a}_{n-a}\ b_{S_n:even}\ (S, d\ (N, n)) \quad (5)$$

$$+ \sum_{n \in s(N)} \sum_{a \in s(N)} S_a\ R^{a}_{n-a}\ b_{S_n:even}\ (S, d\ (N, n)) \quad (6)$$

By Lemma 1.3.i) and ii), we have that

(1)+(4)

$$= \sum_{\substack{a \in s(S) \\ a \notin s(N)}} \{ \sum_{i \in s(S)} R^{i}_{a-i} \sum_{n \in s(N)} R^{a}_{n-a}\ b_{R^{i}_{n-i}:even}\ (d\ (S, i), d\ (N, n))$$

$$+ S_a \sum_{n \in s(N)} R^{a}_{n-a}\ b_{S_n:even}\ (S, d\ (N, n)) \}$$

$$= \sum_{\substack{a \in s(S) \\ a \notin s(N)}} \{ \sum_{i \in s(S)} R^{i}_{a-i} \sum_{\substack{A \cup B \subset N \\ \# A = i(S,i) - 1 \\ \# B = i(S,a) + 1 \\ \# s(B) - \# \{ n \in s(B) \cap s(A) \mid i(A,n) : odd \} : odd}} R^{i}_{A-i}\ R^{a}_{B-a}\ b\ (r\ (S, [i, a]), d\ (N, A \cup B))$$

$$+ S_a \sum_{\substack{A \cup B \subset N \\ \# A = \# N - \# S - 1 \\ \# B = i(S,a) + 1 \\ \# s(B) - \# \{ n \in s(B) \cap s(A) \mid i(A,n) : odd \} : odd}} S_A\ R^{a}_{B-a}\ b\ (r\ (S, a), d\ (N, A \cup B)) \}$$

$$= \sum_{\substack{a \in s(S) \\ a \notin s(N)}} \sum_{\substack{B \subset N \\ \# B = i(S,a) + 1}} R^{a}_{B-a} \{ \sum_{i \in s(S)} \sum_{\substack{A \subset d(N,B) \\ \# A = i(S,i) - 1 \\ \# s(B) - \# \{ n \in s(A) \cap s(B) \mid i(A,n) : odd \} : odd}} R^{i}_{[a \mid A]-i}\ b(r(S,[i, a]), d(N, A \cup B))$$

$$+ \sum_{\substack{C \subset d(N,B) \\ \# C = \# N - \# S - 1 \\ \# s(B) - \# \{ n \in s(B) \cap s(C) \mid i(C,n) : odd \} : odd}} S_{[a \mid C]}\ b\ (r\ (S, a), d\ (N; A \cup B)) \}.$$

By the definition of bracket, we have that

(2)+(5)

$$= \sum_{a \notin s(N) \cup s(S)} \sum_{n \in s(N)} \{ \sum_{i \in s(S)} R^{i}_{a-i}\ R^{a}_{n-a}\ b_{R^{i}_{n-i}:even}\ (d\ (S, i), d\ (N, n))$$

$$+ S_a\ R^{a}_{n-a}\ b_{S_n:even}\ (S, d\ (N, n)) \}$$

$$= \sum_{a \notin s(N) \cup s(S)} \sum_{n \in s(N)} R^{a}_{n-a} \; b\,(S, [\,a \mid d\,(N, n)])$$

$$+ \sum_{a \notin s(N) \cup s(S)} \sum_{n \in s(N)} R^{a}_{n-a} \; \{\sum_{i \in s(S)} R^{i}_{a-i} \; b_{R^{i}_{n-i}:\,odd} \; (d\,(S, i), d\,(N, n))$$

$$+ S_{a} \; b_{S_{n}:\,odd} \; (S, d\,(N, n))\}.$$

By Lemma 1.3.iii) and iv), we have that

(3)+(6)

$$= \sum_{a \in s(N)} \sum_{n \in s(N)} R^{a}_{n-a} \; \{\sum_{i \in s(S)} R^{i}_{a-i} \; b_{R^{i}_{n-i}:\,even} \; (d\,(S, i), d\,(N, n))$$

$$+ S_{a} \; b_{S_{n}:\,even} \; (S, d\,(N, n))\}$$

$$= \sum_{a \in s(N)} \sum_{n \in s(N)} R^{a}_{n-a} \; \{\sum_{\substack{A \cup B \subset S \\ 1 \leq \sharp A \leq i(N,a)+1 \\ \sharp B \leq i(N,n)-1 \\ \sharp s(A) - \sharp\{i \in s(A) \cap s(B) \mid i(B,i):\,odd\}:\,odd}} R^{A}_{a-A} \; R^{B}_{n-B} \; S_{a}^{i(N,a)-\sharp A+1} \; S_{n}^{i(N,n)-\sharp B-1}$$

$$\times bracket\,(d\,(S, A \cup B), r\,(N, [\,a, n]))$$

$$+ \sum_{\substack{A \cup B \subset S \\ \sharp A \leq i(N,a) \\ \sharp B \leq i(N,n)-1 \\ i(N,n)-\sharp B-1\,:\,even}} R^{A}_{a-A} \; R^{B}_{n-B} \; S_{a}^{i(N,a)-\sharp A+1} \; S_{n}^{i(N,n)-\sharp B-1} \; b\,(d\,(S, A \cup B), r\,(N, [\,a, n]))\}$$

$$= \sum_{a \in s(N)} \sum_{n \in s(N)} R^{a}_{n-a} \{\sum_{\substack{A \cup B \subset S \\ \sharp A = i(N,a)+1 \\ \sharp B \leq i(N,n)-1 \\ \sharp s(A) - \sharp\{i \in s(A) \cap s(B) \mid i(B,i):\,odd\}:\,odd}} R^{A}_{a-A} \; R^{B}_{n-B} \; S_{n}^{i(N,n)-\sharp B-1} \; b(d(S, A \cup B), r(N, [\,a, n]))$$

$$+ \sum_{\substack{A \cup A \subset S \\ 1 \leq \sharp A \leq i(N,a) \\ \sharp B \leq i(N,n)-1 \\ \sharp s(A) - \sharp\{i \in s(A) \cap s(B) \mid i(B,i):\,odd\}:\,odd \text{ and } i(N,n)-\sharp B-1\,:\,odd \text{ or} \\ \sharp s(A) - \sharp\{i \in s(A) \cap s(B) \mid i(B,i):\,odd\}:\,even \text{ and } i(N,n)-\sharp B-1\,:\,even}} R^{A}_{a-A} \; R^{B}_{n-B} \; S_{a}^{i(N,a)-\sharp A+1} \; S_{n}^{i(N,n)-\sharp B-1}$$

$$b\,(d\,(S, A \cup B), r\,(N, [\,a, n]))$$

$$+ \sum_{\substack{B \subset S \\ \sharp B \leq i(N,n)-1 \\ i(N,n)-\sharp B-1\,:\,even}} R^{B}_{n-B} \; S_{a}^{i(N,a)+1} \; S_{n}^{i(N,n)-\sharp B-1} \; b\,(d\,(S, A \cup B), r\,(N, [\,a, n]))\}.$$

Then we have the results.

COROLLARY 2.2. *If S and N are both sets, then we have that*

$$d\;bracket\,(S, N)$$

$$= \sum_{a \notin set(N) \cup set(S)} bracket\,([\,a \mid S], [\,a \mid N])$$

$$+ \sum_{a \in N} \sum_{n \in N} R^{a}_{n-a} \; S^{2}_{a} \; bracket\,(S, del\,(N, [\,a, n])).$$

COROLLARY 2.3. *If N is a set and S is a list, then we have that*

$$d\;bracket\,(S, N)$$

$$= \sum_{\substack{a \in set(S) \\ a \notin set(N)}} \sum_{\substack{B \subset N \\ \sharp B = ind(S,a)+1 \\ \sharp B\,:\,odd}} R^{a}_{B-a} \; bracket\,(rest\,(S, a), [\,a \mid del\,(N, B)])$$

$$+ \sum_{a \notin set(N) \cup set(S)} \sum_{n \in N} R^{a}_{n-a} \; bracket\,(S, del\,([\,a \mid N], n))$$

$$+\sum_{a\in N}\sum_{n\in N} R^{a}_{n-a}\ \{\sum_{[i,i]\subset S}(R^{i}_{a-i})^{2}\ \mathit{bracket}\ (\mathit{del}\ (S,[i,i]),\mathit{del}\ (N,[a,n]))$$

$$+S^{2}_{a}\ \mathit{bracket}\ (S,\mathit{del}\ (N,[a,n]))\}.$$

COROLLARY 2.4. *If S is a set and N is a list, then we have that*

$$d\ \mathit{bracket}\ (S,N)$$

$$=\sum_{\substack{a\in S\\ a\notin \mathit{set}(N)}}\{\sum_{[n,n]\subset N}(R^{a}_{n-a})^{2}\sum_{i\in \mathit{set}(S)}R^{i}_{a-i}\ \mathit{bracket}\ (\mathit{del}\ (S,[i,a]),\mathit{del}\ (N,[n,n]))$$

$$+\sum_{\substack{B\subset N\\ \sharp B=2}}R^{a}_{B-a}\sum_{\substack{C\subset \mathit{del}(N,B)\\ \sharp C=\sharp N-\sharp S-1\\ \sharp \mathit{set}(B)-\sharp\{n\in \mathit{set}(B)\cap \mathit{set}(C)\mid \mathit{ind}(C,n):\mathit{odd}\}:\mathit{odd}}}S_{[a\mid C]}\ \mathit{bracket}\ (\mathit{del}\ (S,a),\mathit{del}\ (N,B\cup C))\}$$

$$+\sum_{a\notin S\cup \mathit{set}(N)}\ \sum_{n\in \mathit{set}(N)}R^{a}_{n-a}\ \{\mathit{bracket}\ (S,\mathit{del}\ ([a\mid N],n))$$

$$+S_{a}\ \mathit{bracket}_{S_n:\mathit{odd}}\ (S,\mathit{del}\ (N,n))\}$$

$$+\sum_{a\in s(N)}\sum_{n\in s(N)}R^{a}_{n-a}\ \{\sum_{\substack{A\cup B\subset S\\ \sharp A=i(N,a)+1\\ \sharp s(A):\mathit{odd}\\ \sharp B\leq i(N,n)-1}}R^{A}_{a-A}\ R^{B}_{n-B}\ S_{n}^{i(N,n)-\sharp B-1}\ b\ (d\ (S,A\cup B),r\ (N,[a,n]))$$

$$+\sum_{\substack{A\cup B\subset S\\ 1\leq \sharp A\leq i(N,a)\\ \sharp B\leq i(N,n)-1\\ \sharp s(A):\mathit{odd}\ \mathit{and}\ i(N,n)-\sharp B-1\ :\mathit{odd}\ \mathit{or}\\ \sharp s(A):\mathit{even}\ \mathit{and}\ i(N,n)-\sharp B-1\ :\mathit{even}}}R^{A}_{a-A}\ R^{B}_{n-B}\ S_{a}^{i(N,a)-\sharp A+1}\ S_{n}^{i(N,n)-\sharp B-1}\ b(d(S,A\cup B),r(N,[a,n]))$$

$$+\sum_{\substack{B\subset S\\ \sharp B\leq i(N,n)-1\\ i(N,n)-\sharp B-1\ :\mathit{even}}}R^{B}_{n-B}\ S_{a}^{i(N,a)+1}\ S_{n}^{i(N,n)-\sharp B-1}\ b\ (d\ (S,B),r\ (N,[a,n]))\}.$$

COROLLARY 2.5. *If S and N are list such that ind (S, s) $\leqq 2$ for each item s of S and ind (N, n) $\leqq 2$ for each item n of N, then we have that*

$$d\ \mathit{bracket}\ (S,N)$$

$$=\sum_{\substack{a\in s(S)\\ a\notin s(N)\\ i(S,a)=1}}\{\sum_{[b,b]\subset N}(R^{a}_{b-a})^{2}\ b\ (d\ (S,a),d\ ([a\mid N],[b,b]))$$

$$+\sum_{\substack{[b,b,c]\subset N\\ b\neq c}}\sum_{\substack{i\in s(S)\\ i(S,i)=2}}R^{i}_{a-i}\ R^{i}_{b-i}\ R^{a}_{b-a}\ R^{a}_{c-a}\ b\ (d\ (S,[i,i,a]),d\ (N,[b,b,c]))$$

$$+\sum_{\substack{[b,b,c\mid A]\subset N\\ \sharp A=\sharp N-\sharp S-2\\ b\neq c}}S_{[a,b\mid A]}\ R^{a}_{b-a}\ R^{a}_{c-a}\ b\ (d\ (S,a),d\ (N,[b,b,c\mid A]))\}$$

$$+\sum_{\substack{a\in s(S)\\ a\notin s(N)\\ i(S,a)=2}}\{\sum_{\substack{i\in s(S)\\ i(S,i)=1}}\sum_{\substack{[a,b,c]\subset N\\ \sharp s([a,b,c])=3}}R^{i}_{a-i}\ R^{a}_{a-a}\ R^{a}_{b-a}\ R^{a}_{c-a}\ b\ (d\ (S,[i,a,a]),d\ (N,[a,b,c]))$$

$$+\sum_{\substack{i\in s(S)\\ i(S,i)=2}}\sum_{\substack{[b,b,c,c]\subset N\\ b\neq c}}R^{i}_{a-i}\ b\ ([i,a,a,a],[b,b,c,c])$$

$$\times b\ (d\ (S,[i,i,a,a]),d\ (N,[b,b,c,c]))$$

$$+\sum_{\substack{i\in s(S)\\ i(S,i)=2}}\sum_{\substack{[b,c,d,e]\subset N\\ \sharp s([b,c,d,e])=4}}R^{i}_{a-i}\ b\ ([i,a,a,a],[b,c,d,e])$$

$$\times b\,(d\,(S,[i,i,\alpha,\alpha]),\,d\,(N,[b,c,d,e]))$$

$$+\sum_{\substack{[b,c,d\,|\,A]\subset N\\ \sharp A=\sharp N-\sharp S-1\\ \sharp s([b,c,d])=3}} S_a\ S_A\ R^a_{b-a}\ R^a_{c-a}\ R^a_{d-a}\ b\,(d\,(S,[\alpha,\alpha]),\,d\,(N,[b,c,d\,|\,A]))$$

$$+\sum_{\substack{[b,b,c,c\,|\,A]\subset N\\ \sharp A=\sharp N-\sharp S-2\\ b\neq c}} S_a\ S_A\ b([\alpha,\alpha,\alpha],[b,b,c,c])\,b(d(S,[\alpha,\alpha]),d(N,[b,b,c,c\,|\,A]))$$

$$+\sum_{\substack{[b,b,c,c,d\,|\,A]\subset N\\ \sharp A=\sharp N-\sharp S-3\\ \sharp s([b,c,d])=3}} S_a\ S_{[b,c\,|\,A]}\ R^a_{b-a}\ R^a_{c-a}\ R^a_{d-a}\ b(d(S,[\alpha,\alpha]),d(N,[b,b,c,c,d\,|\,A]))\}$$

$$+\sum_{\substack{a\notin s(S)\\ a\notin s(N)}} \{bracket\,([\alpha\,|\,S],[\alpha\,|\,N])$$

$$+\sum_{\substack{n\in s(N)\\ i(N,n)=2}}\ \sum_{\substack{i\in s(S)\\ i(S,i)=2}} R^a_{n-a}\ R^i_{a-i}\ R^i_{n-i}\ b\,(d\,(S,[i,i]),\,d\,(N,[n,n]))$$

$$+\sum_{\substack{n\in s(N)\\ i(N,n)=2}} R^a_{n-a}\ S_a\ S_n\ b\,(S,\,d\,(N,[n,n]))\}$$

$$+\sum_{\substack{a\in s(N)\\ i(N,a)=1}}\ \sum_{\substack{n\in s(N)\\ i(N,n)=1}} R^a_{n-a}\ \{\sum_{[a,a]\subset S} (R^a_{a-a})^2\ b\,(d\,(S,[a,a]),\,d\,(N,[\alpha,n]))$$

$$+S^2_a\ b\,(S,\,d\,(N,[\alpha,n]))\}$$

$$+\sum_{\substack{a\in s(N)\\ i(N,a)=1}}\ \sum_{\substack{n\in s(N)\\ i(N,n)=2}} R^a_{n-a}\ \{\sum_{[a,a]\subset S} (R^a_{a-a})^2\ b\,(d\,(S,[a,a]),\,d\,(N,[\alpha,n]))$$

$$+\sum_{[a,a]\subset S} R^a_{a-a}\ R^a_{n-a}\ b\,(d\,(S,[a,a]),\,d\,(N,[n,n]))$$

$$+\sum_{a\in S} S_a\ b\,([a],[\alpha,n])\ b\,(d\,(S,[a]),\,d\,(N,[\alpha,n,n]))\}$$

$$+\sum_{\substack{a\in s(N)\\ i(N,a)=2}}\ \sum_{\substack{n\in s(N)\\ i(N,n)=1}} R^a_{n-a}\ \{\sum_{\substack{[a,b,c]\subset S\\ \sharp s([a,b,c])=3}} R^a_{a-a}\ R^b_{a-b}\ R^c_{a-c}\ b\,(d\,(S,[a,b,c]),\,d\,(N,[\alpha,\alpha,n]))$$

$$+\sum_{[a,b]\subset S} R^a_{a-a}\ R^b_{a-b}\ S_a\ b\,(d\,(S,[a,b]),\,d\,(N,[\alpha,\alpha,n]))$$

$$+S^3_a\ b\,(S,\,d\,(N,[\alpha,\alpha,n]))\}$$

$$+\sum_{\substack{a\in s(N)\\ i(N,a)=2}}\ \sum_{\substack{n\in s(N)\\ i(N,n)=2}} R^a_{n-a}\ \{\sum_{\substack{[a,a,b,b]\subset S\\ a\neq b}} b\,([a,a,b,b],[\alpha,\alpha,\alpha,n])$$

$$\times b\,(d\,(S,[a,a,b,b]),\,d\,(N,[\alpha,\alpha,n,n]))$$

$$+\sum_{\substack{[a,b,c,d]\subset S\\ \sharp s([a,b,c,d])=4}} b([a,b,c,d],[\alpha,\alpha,\alpha,n])\,b(d(S,[a,b,c,d]),d(N,[\alpha,\alpha,n,n]))$$

$$+\sum_{\substack{[a,b,c]\subset S\\ \sharp s([a,b,c])=3}} b\,([a,b,c],[\alpha,\alpha,\alpha,n])\ b\,(d\,(S,[a,b,c]),\,d\,(N,[\alpha,\alpha,n,n]))$$

$$+\sum_{[a,a]\subset S} b\,([a,a],[\alpha,\alpha,n])\ S_a\ b\,(d\,(S,[a,a]),\,d\,(N,[\alpha,\alpha,n,n]))$$

$$+\sum_{a\in S} b\,([a],[\alpha,n])\ S^2_a\ b\,(d\,(S,[a]),\,d\,(N,[\alpha,\alpha,n,n]))\}$$

3. Some relations of H*(X)

In this section we apply Theorem 2.1 and its corollaries to pick up systematically the defining relations of $H^*(X)$.

We first quote the defining relations proved in [7 : Theorem 3,5] and [6 : Theorem 2.2, 3.1, 4.1, 5.1]. These are directly drived from the results in section 2. So we omit those proofs.

THEOREM **3.1.** *Let* $S=[i_0, i_1,\cdots,i_{n+2}]$ *and* $N=[j_1, j_2,\cdots,j_{n+1}]$ *with* $n\geqq 0$, $i_0+4\leqq m\leqq i_0+2n+4$, $0\leqq i_0<i_1<\cdots<i_{n+2}$, $S\cap N=[\,]$, $S\cup N=[i_0, i_0+1, i_0+2,\cdots, i_0+2n+3]$ *and* $i_p\leqq i_0+2(p-1)-1$ *if* $i_p\geqq m$, $i_{n+2}\leqq i_0+2n+2$ *if* $m=i_0+2n+4$ *and* $i_{n+2}\leqq i_0+2n+1$ *if* $m\leqq i_0+2n+3$, *in addition to being* $i_p\leqq i_0+2p-1$ *for each p such that* $1\leqq p\leqq n+2$. *Then we have that*

$$d\ bracket\ (S, [m, m \mid N])$$

$$=\sum_{s\leq p\leq r} (R^{i_p}_{m-i_p})^2\ bracket\ (del\ (S, i_p), [i_p \mid N]),$$

where r is the largest index p such that $i_p<m$ $(1\leqq p\leqq n+2)$ *and s is the largest index p such that* $i_p=i_0+2p-1$ $(1\leqq p\leqq r)$.

THEOREM **3.2.** *Let* $S=[i_0, i_1,\cdots, i_n]$ *and* $N=[j_1, j_2,\cdots, j_n]$ *with* $n\geqq 1$, $3\leqq m\leqq 2n+1$, $0=i_0<i_1<i_2<\cdots<i_n$, $S\cap N=[\,]$, $S\cup N=[0, 1, 2,\cdots, 2n]$ *and* $i_p\leqq 2(p-1)$ *if* $i_p\geqq m$, $i_n\leqq 2n-1$ *if* $m=2n+1$ *and* $i_n\leqq 2n-2$ *if* $m\leqq 2n$, *in addition to being* $i_p\leqq 2p$ *for each index p. Then we have that*

$$d\ bracket\ (S, [m, m \mid N])$$

$$=\sum_{s\leq p\leq r} (R^{i_p}_{m-i_p})^2\ bracket\ (del\ (S, i_p), [i_p \mid N]),$$

where r is the largest index p such that $i_p<m$ *and s is the largest index p such that* $i_p=2p$ $(0\leqq p\leqq r)$.

THEOREM **3.3.** *Let* $S=[i_0, i_1,\cdots, i_n]$ *and* $N=[j_1, j_2,\cdots, j_{n+3}]$ *satisfying that* $n\geqq 0$, $0=i_0<i_1<\cdots<i_n$, $S\cap N=[\,]$, $S\cup N=[0, 1, 2,\cdots, 2n+3]$ *and* $i_k\leqq 2k+1$ *for each* $1\leqq k\leqq n$. *Then we have that*

$$d\ bracket\ (S, N)$$
$$=\sum_{k\in N} S^2_k\ bracket\ ([k \mid S], del\ (N, k)).$$

If there exist an index p such that $i_p=2p+1$ *then the summation is restricted as* $\sum_{k\in N,\ k<2q+1}$, *where q is the least index p such that* $i_p=2p+1$.

THEOREM **3.4.** *Let* $S=[i_0, i_1,\cdots, i_n]$ *and* $N=[j_1, j_2,\cdots, j_{n+4}]$ *satisfying that* $n\geqq 0$, $0=i_0<i_1<\cdots<i_n$, $S\cap N=[\,]$, $S\cup N=[0, 1, 2,\cdots, 2n+4]$ *and* $i_k\leqq 2k+2$ *for each* $1\leqq k\leqq n$.

Then we have that

$$d\ bracket\ (S, N) = \sum_{k\in N} S^2_k\ bracket\ ([k \mid S], del\ (N, k)).$$

If there exist an index p such that $i_p = 2p+2$ *then the summation is restricted as* $\sum_{\substack{k\in N\\ k<2q+2}}$, *where q is the least index p such that* $i_p = 2p+2$.

NOTATIONS. If x is any real number, we write $\lceil x \rceil$ = the least integer greater than or equal to x (the "ceiling" of x). This notation is used in [3].

THEOREM 3.5. *Let* $k \geqq -1$, $i \geqq max\ \{k, 0\}$ *and* $n \geqq \lceil (k+1)/2 \rceil$. *Let* $S = [i_0, i_1, \cdots, i_n]$ *and* $N = [j_1, j_2, \cdots, j_{n+3}]$ *satisfying that* $i_0 < i_1 < \cdots < i_n$, $S \cap N = [\,]$, $S \cup N = [i-k, i-k+1, i-k+2, \cdots, i-k+2n+3]$, $i_0 = i-k$, $i_q \leqq i-k+2q-1$ *if* $1 \leqq q \leqq \lceil (k+1)/2 \rceil$, $i_q \leqq i-k+2q+1$ *if* $\lceil (k+1)/2 \rceil < q \leqq n$. *Then we have that*

$$d\ bracket\ ([i, i \mid S], N) = \sum_{\substack{i<a<i+2n-k+3\\ a\notin S}} (R^i_{a-i})^2\ bracket\ ([a \mid S], del\ (N, a)).$$

If there exist an index q such that $i_q = i-k+2q+1$ *then the summation is restricted as* $\sum_{\substack{i<a<i-k+2p+1\\ a\notin S}}$, *where p is the least index q such that* $i_q = i-k+2q+1$.

THEOREM 3.6. *Let* $S = [i_0, i_1, \cdots, i_n]$ *and* $N = [j_1, j_2, \cdots, j_{n+4}]$ *satisfying that* $k \geqq 0$, $n \geqq \lceil k/2 \rceil$, $0 = i_1 < i_2 < \cdots < i_n$, $S \cap N = [\,]$, $S \cup N = [0, 1, 2, \cdots, 2n+4]$ *and* $i_q \leqq 2q$ *if* $0 \leqq q \leqq \lceil k/2 \rceil$, $i_q \leqq 2q+2$ *if* $\lceil k/2 \rceil < q \leqq n$. *Then we have that*

$$d\ bracket\ ([k, k \mid S], N) = \sum_{\substack{k<a<2n+4\\ a\notin S}} (R^k_{a-k})^2\ bracket\ ([a \mid S], del\ (N, a)).$$

If there exist an index q such that $i_q = 2q+2$ *then the summation is restricted as* $\sum_{\substack{k<a<2p+2\\ a\notin S}}$, *where p is the least index q such that* $i_q = 2q+2$.

Now we shall discuss the defining relations of the form $g(S)\ g(S') = \cdots$. The first Theorem is proved in [6 : Theorem 2.1].

THEOREM 3.7. *Let* $S = [i_0, i_1, \cdots, i_n]$ *and* $N = [j_0, j_1, \cdots, j_{n+1}]$ *satisfying that* $n \geqq 0$, $0 = i_0 < i_1 < \cdots < i_n$, $i_k \leqq 2k$ *for each* $1 \leqq k \leqq n$, $S \cap N = [\,]$ *and* $S \cup N = [0, 1, 2, \cdots, 2n+2]$. *Then we have that*

$$d\ bracket\ (del\ (S, 0), N) = S_0\ bracket\ (S, N) + \sum_{k\in N} S^2_k\ bracket\ (del\ ([k \mid S], 0), del\ (N, k)).$$

If there exist an index p such that $i_p = 2p$ *then the summation is restricted as* $\sum_{\substack{k\in N\\ k<2p}}$, *where p is the least index q such that* $i_q = 2q$.

PROOF. Since *del* $(S,\ 0)$ and N are both sets with *del* $(S,\ 0)\cap N=[\ ]$ and *del* $(S,\ 0)\cup N=[1,\ 2,\ 3,\cdots,\ 2n+2]$, we have, by Corollary 2.2 and Lemma 1.2. i) and iii), that

$$\begin{aligned} &d\ bracket\ (del\ (S,\ 0),\ N)\\ &=bracket\ ([0\mid del\ (S,\ 0)],\ [0\mid N])\\ &+\sum_{a\in N}\sum_{n\in N} R^{a}_{n-a}\ S^{2}_{a}\ bracket\ (del\ (S,\ 0),\ del\ (N,\ [a,\ n]))\\ &=S_0\ bracket\ (S,\ N)+\sum_{a\in N} S^{2}_{a}\ bracket\ ([a\mid del\ (S,\ 0)],\ del\ (N,\ a)). \end{aligned}$$

Next we assume that there exist an index q such that $i_q=2q$ and p is the least index q such that $i_q=2q$. If $k\in N$ with $k\geqq 2p$ then *del* $([k\mid S],\ 0)$ contains $p-1$ items $i_1,\ i_2,\cdots,\ i_{p-1}$ less than $2p$ and *del* $(N,\ k)$ contains p items $j_0,\ j_1,\cdots,\ j_{p-1}$ less than $2p$. Since $\#$*del* $([k\mid S],\ 0)=\#$*del* $(N,\ k)$, by the definition of bracket, $R^{i_r}_{j_q-i_r}$ with $j_q<i_r$ must be appear for each monomial. So in this case, we have *bracket* (*del* $([k\mid S],\ 0)$, *del* $(N,\ k))=0$.

THEOREM 3.8. *Let* $S=[i_0,\ i_1,\cdots,\ i_n]$ *and* $N=[j_0,\ j_1,\cdots,\ j_{n+1}]$ *satisfying that* $n\geqq 1$, $i_0=0$, $i_1=2$, $i_{k-1}<i_k\leqq 2k$ *for each* $1\leqq k\leqq n$, $S\cap N=[\ ]$ *and* $S\cup N=[0,\ 1,\ 2,\cdots,\ 2n+2]$. *Then we have that*

$$\begin{aligned} &d\ bracket\ (del\ ([0\mid S],\ 2),\ [1\mid N])\\ &=bracket\ ([0],\ [1,\ 2])\ bracket\ (S,\ N)\\ &+\sum_{a\in del(N,1)}\{(R^{0}_{a})^{2}\ S^{2}_{1}+(R^{0}_{1})^{2}\ S^{2}_{a}\}\ bracket\ (del\ ([a\mid S],\ [0,\ 2]),\ del\ (N,\ [1,\ a])). \end{aligned}$$

PROOF. By Corollary 2.5, we have that

$$\begin{aligned} &d\ bracket\ (del\ ([0\mid S],\ 2),\ [1\mid N])\\ &=bracket\ ([0\mid S],\ [1,2\mid N])\\ &+\sum_{\substack{a,\beta\in del(N,1)\\ a<\beta}} R^{a}_{\beta-a}\ \{(R^{0}_{a})^{2}\ bracket\ (del\ (S,\ [0,\ 2]),\ del\ ([1\mid N],\ [a,\beta]))\\ &\qquad +S^{2}_{a}\ bracket\ (del\ ([0\mid S],\ 2),\ del\ ([1\mid N],\ [a,\beta]))\} \end{aligned}$$

By Lemma 1.2, we have that

$$\begin{aligned} &=(R^{0}_{1})^{2}\ bracket\ (del\ (S,\ 0),\ del\ ([2\mid N],\ 1))\\ &+R^{0}_{1}\ S_1\ bracket\ (S,\ del\ ([2\mid N],\ 1))\\ &+S^{2}_{1}\ bracket\ ([0\mid S],\ del\ ([2\mid N],\ 1))\\ &+\sum_{\substack{a,\beta\in del(N,1)\\ a<\beta}} (R^{0}_{a})^{2}\ S^{2}_{1}\ R^{a}_{\beta-a}\ bracket\ (del\ (S,\ [0,\ 2]),\ del\ (N,\ [1,a,\beta]))\\ &+\sum_{\substack{a,\beta\in del(N,1)\\ a<\beta}} (R^{0}_{1})^{2}\ S^{2}_{a}\ R^{a}_{\beta-a}\ bracket\ (del\ (S,\ [0,\ 2]),\ del\ (N,\ [1,a,\beta]))\\ &=(R^{0}_{1})^{2}\ S_2\ bracket(del(S,0),\ del(N,1))+R^{0}_{1}\ R^{0}_{2}\ S_1\ bracket(del(S,0),\ del(N,1))\\ &+R^{0}_{1}\ S_1\ S_2\ bracket\ (S,\ del\ (N,\ 1))+S^{2}_{1}\ R^{0}_{2}\ bracket\ (S,\ del\ (N,\ 1)) \end{aligned}$$

$$+\sum_{\alpha\in del(N,1)} S_1^2\ (R_\alpha^0)^2\ \ bracket\ (del\ ([\alpha\mid S],[0,2]),\ del\ (N,[1,\alpha]))$$

$$+\sum_{\alpha\in del(N,1)} (R_1^0)^2\ S_\alpha^2\ \ bracket\ (del\ ([\alpha\mid S],[0,2]),\ del\ (N,[1,\alpha]))$$

By Lemma 1.2. iii), we have that

$$bracket\ ([0],[1,2])\ bracket\ (S,N)$$

$$= R_1^0\ S_2\ \ bracket\ (S,N) + R_2^0\ S_1\ \ bracket\ (S,N)$$

$$= (R_1^0)^2\ S_2\ \ bracket\ (del\ (S,0),\ del\ (N,1)) + R_1^0\ S_1\ S_2\ \ bracket\ (S,\ del\ (N,1))$$

$$+ R_1^0\ R_2^0\ S_1\ \ bracket\ (del\ (S,0),\ del\ (N,1)) + S_1^2\ R_2^0\ \ bracket\ (S,\ del\ (N,1)).$$

Then we have the result.

THEOREM 3.9. *Let* $S=[i_0, i_1,\cdots, i_n]$ *and* $N=[j_0, j_1,\cdots, j_{n+1}]$ *satisfying that* $n\geqq 1$, $i_0=0$, $i_1=1$, $i_{k-1}<i_k\leqq 2k$ *for each* $1\leqq k\leqq n$, $S\cap N=[\,]$ *and* $S\cup N=[0,1,2,\cdots,2n+2]$ *and* $i_2=3$ *if* $n\geqq 2$. *Then we have that*

$$d\ bracket\ (del\ ([0\mid S],1),[2\mid N])$$

$$= bracket\ ([0],[1,2])\ bracket\ (S,N)$$

$$+\sum_{\alpha\in del(N,2)} \{(R_\alpha^0)^2\ S_2^2 + (R_2^0)^2\ S_\alpha^2\}\ bracket\ (del\ ([\alpha\mid S],[0,1]),\ del\ (N,[2,\alpha])).$$

PROOF. This is essentially same as above one. So we omit it.

THEOREM 3.10. *Let* $S=[i_0, i_1,\cdots, i_n]$ *and* $N=[j_0, j_1,\cdots, j_{n+1}]$ *satisfying that* $n\geqq 2$, $i_0=0$, $i_1=1$, $i_2=2$, $i_{k-1}<i_k\leqq 2k$ *for each* $1\leqq k\leqq n$, $S\cap N=[\,]$ *and* $S\cup N=[0,1,2,\cdots,2n+2]$. *Then we have that*

$$d\ \{R_1^0\ \ bracket\ (del\ (S,2),N) + R_2^0\ \ bracket\ (del\ (S,1),N)$$

$$+\sum_{\alpha\in N} S_\alpha^2\ \ bracket\ (del\ ([0,\alpha\mid S],[1,2]),\ del\ (N,\alpha))\}$$

$$= bracket\ ([0],[1,2])\ bracket\ (S,N)$$

$$+\sum_{\alpha\in N}\ \sum_{\beta\in del(N,\alpha)} S_\alpha^2\ (R_\beta^0)^2\ \ bracket\ (del\ ([\alpha,\beta\mid S],[0,1,2]),\ del\ (N,[\alpha,\beta])).$$

PROOF. By Corollary 2.5, we have that

$$d\ \{R_1^0\ \ bracket\ (del\ (S,2),N) + R_2^0\ \ bracket\ (del\ (S,1),N)$$

$$+\sum_{\alpha\in N} S_\alpha^2\ \ bracket\ (del\ ([0,\alpha\mid S],[1,2]),\ del\ (N,\alpha))\}$$

$$= R_1^0\ \ \{bracket\ (S,[2\mid N]) + \sum_{\alpha\in N}\ \sum_{\beta\in N} S_\alpha^2\ R_{\beta-\alpha}^\alpha\ \ bracket\ (del\ (S,2),\ del\ (N,[\alpha,\beta]))\}$$

$$+ R_1^0\ \ R_1^1\ \ bracket\ (del\ (S,1),N)$$

$$+ R_2^0\ \ \{bracket\ (S,[1\mid N]) + \sum_{\alpha\in N}\ \sum_{\beta\in N} S_\alpha^2\ R_{\beta-\alpha}^\alpha\ \ bracket\ (del\ (S,1),\ del\ (N,[\alpha,\beta]))\}$$

$+\sum_{\alpha\in N} S^2_\alpha$ $\{bracket\ (del\ ([\,0\,,\alpha\mid S],\ 2\,),\ del\ ([\,1\mid N],\ \alpha))$

$+\ bracket\ (del\ ([\,0\,,\alpha\mid S],\ 1\,),\ del\ ([\,2\mid N],\ \alpha))$

$+\sum_{\beta,\gamma\in del(N,\alpha)} R^{\beta}_{\gamma-\beta}\ (R^0_\beta)^2\ \ bracket\ (del\ ([\,\alpha\mid S],\ [\,0\,,1\,,2\,]),\ del\ (N,\ [\,\alpha,\beta,\gamma\,]))\}$

$= R^0_1\ R^0_2\ bracket(del(S,0\,),N) + R^0_1\ R^1_1\ bracket(del(S,1\,),N) + R^0_1\ S_2\ \ bracket(S,\ N)$

$+\sum_{\alpha\in N} S^2_\alpha\ R^0_1\ \ bracket\ (del\ ([\,\alpha\mid S],\ 2\,),\ del\ (N,\alpha)) + R^0_1\ R^1_1\ \ bracret\ (del\ (S,1\,),\ N)$

$+ R^0_2\ R^0_1\ \ bracret\ (del\ (S,0\,),\ N) + R^0_2\ S_1\ \ bracket\ (S,\ N)$

$+\sum_{\alpha\in N} S^2_\alpha\ R^0_2\ \ bracket\ (del\ ([\,\alpha\mid S],\ 1\,),\ del\ (N,\alpha))$

$+\sum_{\alpha\in N} S^2_\alpha\ R^0_1\ \ bracket\ (del\ ([\,\alpha\mid S],\ 2\,),\ del\ (N,\alpha))$

$+\sum_{\alpha\in N} S^2_\alpha\ R^0_2\ \ bracket\ (del\ ([\,\alpha\mid S],\ 1\,),\ del\ (N,\alpha))$

$+\sum_{\alpha\in N}\ \sum_{\beta\in del(N,\alpha)} S^2_\alpha\ (R^0_\beta)^2\ \ bracket\ (del\ ([\,\alpha,\beta\mid S],\ [\,0\,,1\,,2\,]),\ del\ (N,\ [\,\alpha,\beta\,]))$

$= bracket\ ([\,0\,],\ [\,1\,,2\,])\ bracket\ (S,\ N)$

$+\sum_{\alpha\in N}\ \sum_{\beta\in del(N,\alpha)} S^2_\alpha\ (R^0_\beta)^2\ \ bracket\ (del\ ([\,\alpha,\beta\mid S],\ [\,0\,,1\,,2\,]),\ del\ (N,\ [\,\alpha,\beta\,]))$

Then we have the result.

Next we shall discuss the defining relations of the form $g(S)\ h_i(S') = \cdots$.

THEOREM **3.11.** *Let* $S_n = [i_1,\ i_2,\cdots,\ i_n]$ $(S_0 = [\,])$ *and* $N_n = [j_0,\ j_1,\cdots,\ j_n]$ *satisfying that* $n\geqq 0$, $0 = i_1 < i_2 < \cdots < i_n$, $S_n\cap N_n = [\,]$, $S_n\cup N_n = [\,0\,,1\,,2\,,\cdots,\ 2n]$ *and* $i_k\leqq 2\,(k-1\,)$ *for each* $1\leqq k\leqq n$. *And let* $S' = [i'_0,\ i'_1,\cdots,\ i'_m]$ *and* $N' = [j'_0,\ j'_1,\cdots,\ j'_m]$ *satisfying that* $m\geqq 0$, $2n = i'_0 < i'_1 < \cdots < i'_m$, $S'\cap N' = [\,]$, $S'\cup N' = [\,2n,\ 2n+1\,,\ 2n+2\,,\cdots,\ 2n+2m+1\,]$ *and* $i'_k\leqq i'_0 + 2k-1$ *for each* $1\leqq k\leqq m$. *Then we have that*

$$d\ bracket\ (S_n\cup del\ (S',\ i'_0),\ del\ (N_n,\ 2n)\cup N')$$

$$= bracket\ (S_n,\ N_n)\ bracket\ (S',\ N').$$

PROOF. Since $S_n\cup del\,(S',i'_0)$ and $del\ (N_n,\ 2n)\cup N'$ are both sets with $(S_n\cup del\,(S',\ i'_0))\cap(del\ (N_n,\ 2n)\cup N') = [\,]$ and $(S_n\cup del\ (S',\ i'_0))\cup(del\ (N_n,\ 2n)\cup N') = del\ ([\,0\,,1\,,2\,,\cdots,\ 2n+2m+1\,],\ 2n)$, we have , by Corollary 2.2, that

$$d\ bracket\ (S_n\cup del\ (S',\ i'_0),\ del\ (N_n,\ 2n)\cup N')$$

$$= \sum_{p\in N'} R^{2n}_{p-2n}\ \ bracket\ (S_n\cup del\ (S',\ i'_0),\ N_n\cup del\ (N',\ p))$$

$$= bracket\ (S_n,\ N_n)\ \sum_{p\in N'} R^{2n}_{p-2n}\ \ bracket\ (del\ (S',\ i'_0),\ del\ (N',\ p))$$

$$= bracket\ (S_n,\ N_n)\ bracket\ (S',\ N').$$

THEOREM **3.12.** *Let* $S = [i_0,\ i_1,\cdots,\ i_n]$ *and* $N = [j_1,\ j_2,\cdots,\ j_n]$ *satisfying that* $n\geqq 0$, $0 = i_0 < i_1 < \cdots < i_n$, $S\cap N = [\,]$, $S\cup N = [\,0\,,1\,,2\,,\cdots,\ 2n]$ *and* $i_k\leqq 2k$ *for each* $1\leqq k\leqq n$. *Then we* *have that*

$$d\ bracket\ (S, [2n+2,\ 2n+2 \mid N])$$

$$= R_1^{2n+1}\ bracket\ (S, [2n+1,\ 2n+2 \mid N])$$

$$= \sum_{s \leq p \leq n} (R_{2n-i_p+2}^{i_p})^2\ bracket\ (del\ (S, i_p), [i_p \mid N]),$$

where s is the largest index p such that $i_p = 2p$ $(0 \leqq p \leqq n)$.

PROOF. By Corollary 2.4, we have that

$$d\ bracket\ (S, [2n+2,\ 2n+2 \mid N])$$

$$= \sum_{0 \leq p \leq n} (R_{2n-i_p+2}^{i_p})^2\ \{\sum_{0 \leq k \leq n} R_{i_p-i_k}^{i_k}\ bracket\ (del\ (S, [i_p, i_k]), N)$$

$$+ S_{i_p}\ bracket\ (del\ (S, i_p), N)\}$$

$$+ R_1^{2n+1}\ bracket\ (S, [2n+1,\ 2n+2 \mid N])$$

$$= R_1^{2n+1}\ bracket\ (S, [2n+1,\ 2n+2 \mid N])$$

$$+ \sum_{0 \leq p \leq n} (R_{2n-i_p+2}^{i_p})^2\ bracket\ (del\ (S, i_p), [i_p \mid N]).$$

If $p < s$ then $del\ (S, i_p)$ contains s items $i_0, \cdots, i_{p-1}, i_{p+1}, \cdots, i_s$ less than or equal to $2s$ and $[i_p \mid N]$ contains $s+1$ items $j_1, \cdots, j_s$ and i_p less than $2s$. So we have the results.

REMARK. This is proved in [7 : Theorem 4].

THEOREM 3.13. *Let* $S = [i_0, i_1, \cdots, i_{n-1}]$ *and* $N = [j_0, j_1, \cdots, j_n]$ *satisfying* $n \geqq 1$, $i_0 = 0$, $i_{n-1} = 2n-2$, $i_{k-1} < i_k \leqq 2k$ *for each* $1 \leqq k \leqq n-1$, $S \cap N = []$ *and* $S \cup N = [0, 1, 2, \cdots, 2n]$. *And let* $S' = [i'_0, i'_1, \cdots, i'_m]$ *and* $N' = [j'_0, j'_1, \cdots, j'_m]$ *satisfying that* $m \geqq 1$, $i'_0 = 2n-1$, $i'_1 = 2n$, $i'_{k-1} < i'_k \leqq i'_0 + 2k - 1$ *for each* $1 \leqq k \leqq m$, $S' \cap N' = []$ *and* $S' \cup N' = [2n-1,\ 2n,\ 2n+1, \cdots,\ 2n+2m]$. *Then we have that*

$$d\ \{R_1^{2n-2}\ bracket\ (del\ (S,\ 2n-2) \cup del\ (S',\ 2n), del\ (N, [2n-1,\ 2n]) \cup N')$$

$$+ R_2^{2n-2}\ bracket\ (del\ (S,\ 2n-2) \cup del\ (S',\ 2n-1), del\ (N, [2n-1,\ 2n]) \cup N')$$

$$+ bracket\ ([2n-2 \mid S] \cup del(S', [2n-1, 2n]), del([2n-2 \mid N], [2n-1, 2n]) \cup N')\}$$

$$= bracket\ (S, N)\ bracket\ (S', N')$$

$$+ \sum_{a \in N'} (R_{a-2n+2}^{2n-2})^2\ bracket\ (del\ (S,\ 2n-2) \cup [a \mid del\ (S', [2n-1,\ 2n])],$$

$$del\ ([2n-2 \mid N], [2n-1,\ 2n]) \cup del\ (N', a)).$$

PROOF. By Corollary 2.2, we have that

$$d\ \{R_1^{2n-2}\ bracket\ (del\ (S,\ 2n-2) \cup del\ (S',\ 2n), del\ (N, [2n-1,\ 2n]) \cup N')$$

$$+ R_2^{2n-2}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup \mathit{del}\,(S',\ 2n-1),\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ \mathit{bracket}\,([2n-2\mid S]\cup \mathit{del}\,(S',\ [2n-1,\ 2n]),\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')\}$$

$$= R_1^{2n-2}\ \{\mathit{bracket}\,(S\cup \mathit{del}\,(S',\ 2n),\ \mathit{del}\,([2n-2\mid N],\ [2n-1,\ 2n])\cup N')$$

$$+ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup S',\ \mathit{del}\,(N,\ 2n-1)\cup N')\}$$

$$+ R_1^{2n-2}\ R_1^{2n-1}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup \mathit{del}\,(S',\ 2n-1),\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ R_2^{2n-2}\ \{\mathit{bracket}\,(S\cup \mathit{del}\,(S',\ 2n-1),\ \mathit{del}\,([2n-2\mid N],\ [2n-1,\ 2n])\cup N')$$

$$+ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup S',\ \mathit{del}\,(N,\ 2n)\cup N')\}$$

$$+ \mathit{bracket}\,([2n-2\mid S]\cup \mathit{del}\,(S',\ 2n),\ \mathit{del}\,([2n-2\mid N],\ 2n)\cup N')$$

$$+ \mathit{bracket}\,([2n-2\mid S]\cup \mathit{del}\,(S',\ 2n-1),\ \mathit{del}\,([2n-2\mid N],\ 2n-1)\cup N')$$

$$+ \sum_{a\in N'} (R_{a-2n+2}^{2n-2})^2\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup[a\mid \mathit{del}\,(S',\ [2n-1,\ 2n])],$$

$$\mathit{del}\,([2n-2\mid N],\ [2n-1,\ 2n])\cup \mathit{del}\,(N',\ a)).$$

By Lemma 1.2, we have that

$$R_1^{2n-2}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup S',\ \mathit{del}\,(N,\ 2n-1)\cup N')$$

$$+ R_1^{2n-2}\ R_1^{2n-1}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup \mathit{del}\,(S',\ 2n-1),\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ R_2^{2n-2}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup S',\ \mathit{del}\,(N,\ 2n)\cup N')$$

$$= R_1^{2n-2} \sum_{a\in \mathit{del}(S,\,2n-2)} R_{2n-a}^{a}\ \mathit{bracket}(\mathit{del}(S,[a,\,2n-2])\cup S',\ \mathit{del}(N,[2n-1,\,2n])\cup N')$$

$$+ R_1^{2n-2}\ S_{2n}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup S',\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ \sum_{a\in \mathit{del}(S,\,2n-2)} R_2^{2n-2}\ R_{2n-a-1}^{a}\ \mathit{bracket}(\mathit{del}(S,[a,\,2n-2])\cup S',\ \mathit{del}(N,[2n-1,\,2n])\cup N')$$

$$+ R_2^{2n-2}\ S_{2n-1}\ \mathit{bracket}\,(\mathit{del}\,(S,\ 2n-2)\cup S',\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

This is an expansion of *bracket* (S, N) *bracket* (S', N') with respect to $2n-1$ and $2n$. Similarly, we have that

$$R_1^{2n-2}\ \mathit{bracket}\,(S\cup \mathit{del}\,(S',\ 2n),\ \mathit{del}\,([2n-2\mid N],\ [2n-1,\ 2n])\cup N')$$

$$+ \mathit{bracket}\,([2n-2\mid S]\cup \mathit{del}\,(S',\ 2n),\ \mathit{del}\,([2n-2\mid N],\ 2n)\cup N')$$

$$= \sum_{a\in S} R_1^{2n-1}\ R_{2n-a-2}^{a}\ \mathit{bracket}\,(\mathit{del}\,(S,a)\cup \mathit{del}\,(S',\ 2n),\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ R_1^{2n-2}\ S_{2n-2}\ \mathit{bracket}\,(S\cup \mathit{del}\,(S',\ 2n),\ \mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ \sum_{[i,j]\subset S} R_{2n-i-2}^{i}\ R_{2n-j-1}^{j}\ \mathit{bracket}\,(\mathit{del}\,([2n-2\mid S],\ [i,\ j])\cup \mathit{del}\,(S',\ 2n),$$

$$\mathit{del}\,(N,\ [2n-1,\ 2n])\cup N')$$

$$+ \sum_{i\in S} S_{2n-2}\ R_{2n-i-1}^{i}\ \mathit{bracket}(\mathit{del}([2n-2\mid S],i)\cup \mathit{del}(S',2n),\mathit{del}(N,[2n-1,\,2n])\cup N')$$

$+\sum_{i\in S} S_{2n-1}\ R^{i}_{2n-i-2}\ bracket(del([2n-2 \mid S],i)\cup del(S',2n),del(N,[2n-1,2n])\cup N')$

$=0$

and that

$R^{2n-2}_{2}\ bracket\ (S\cup del\ (S',\ 2n-1),\ del\ ([2n-2 \mid N],[2n-1,\ 2n])\cup N')$

$+ bracket\ ([2n-2 \mid S]\cup del\ (S',\ 2n-1),\ del\ ([2n-2 \mid N],\ 2n-1)\cup N')$

$=\sum_{i\in S} R^{2n-2}_{2}\ R^{i}_{2n-i-2}\ bracket\ (del(S,i)\cup del(S',\ 2n-1),\ del\ (N,[2n-1,\ 2n])\cup N')$

$+ R^{2n-2}_{2}\ S_{2n-2}\ bracket\ (S\cup del\ (S',\ 2n-1),\ del\ (N,\ [2n-1,\ 2n])\cup N')$

$+\sum_{[i,j]\subset S} R^{i}_{2n-i-2}\ R^{j}_{2n-j}\ bracket\ (del\ ([2n-2 \mid S],\ [i,j])\cup del\ (S',\ 2n-1),$

$del\ (N,\ [2n-1,\ 2n])\cup N')$

$+\sum_{i\in S} S_{2n-2}\ R^{i}_{2n-i}\ bracket(del([2n-2 \mid S],i)\cup del(S',2n-1),del(N,[2n-1,2n])\cup N')$

$+\sum_{i\in S} S_{2n}\ R^{i}_{2n-i-2}\ bracket(del([2n-2 \mid S],i)\cup del(S',2n-1),del(N,[2n-1,2n])\cup N')$

$=0.$

Then we have the results.

THEOREM 3.14. *Let* $S=[i_0,i_1,\cdots,i_{n-1}]$ *and* $N=[j_0,j_1,\cdots,j_n]$ *satisfying that* $n\geqq 1$, $i_0=0$, $i_{n-1}=2n-2$, $i_{k-1}<i_k\leqq 2k$ *for each* $1\leqq k\leqq n-1$, $S\cap N=[\,]$ *and* $S\cup N=[0,1,2,\cdots,2n]$. *And let* $S'=[i'_0,i'_1,\cdots,i'_m]$ *and* $N'=[j'_0,j'_1,\cdots,j'_m]$ *satisfying that* $m\geqq 1$, $i'_0=2n-2$, $i'_1=2n-1$, $i'_{k-1}<i'_k\leqq i'_0+2k-1$ *for each* $1\leqq k\leqq m$, $S'\cap N'=[\,]$, $S'\cup N'=[2n-2,\ 2n-1,\ 2n,\cdots,\ 2n+2m-1]$ *and if* $m\geqq 2$ *then* $i'_2>2n$. *Then we have that*

$d\ \{R^{2n-2}_{2}\ bracket\ (del\ (S,\ 2n-2)\cup del\ (S',\ 2n-1),\ del\ (N,[2n-1,\ 2n])\cup N')\}$

$= bracket\ (S,N)\ bracket\ (S',N').$

PROOF. By Corollary 2.2, we have that

$d\ \{R^{2n-2}_{2}\ bracket\ (del\ (S,\ 2n-2)\cup del\ (S',\ 2n-1),\ del\ (N,[2n-1,\ 2n])\cup N')\}$

$= R^{2n-2}_{1}\ R^{2n-1}_{1}\ bracket\ (del(S,\ 2n-2)\cup del(S',\ 2n-1),\ del\ (N,[2n-1,\ 2n])\cup N')$

$+ R^{2n-2}_{2}\ bracket\ (del\ (S,\ 2n-2)\cup S',\ del\ (N,\ 2n)\cup N')$

By Lemma 1.2, we have that

$=\sum_{i\in del(S,2n-2)} R^{2n-2}_{1}\ R^{2n-1}_{1}\ R^{i}_{2n-i}\ bracket\ (del\ (S,[i,\ 2n-2])\cup del\ (S',\ 2n-1),$

$del\ (N,\ [2n-1,\ 2n])\cup del\ (N',\ 2n))$

$+ R^{2n-2}_{1}\ R^{2n-1}_{1}\ R^{2n-2}_{2}\ bracket\ (del\ (S,\ 2n-2)\cup del\ (S',[2n-2,\ 2n-1]),$

$del\ (N,\ [2n-1,\ 2n])\cup del\ (N',\ 2n))$

$+ R_1^{2n-2}\ R_1^{2n-1}\ S_{2n}$ *bracket* $(del(S,\ 2n-2)\cup del(S',\ 2n-1),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ \sum_{i\in del(S, 2n-2)} R_2^{2n-2}\ R^i_{2n-i-1}\ R_2^{2n-2}$ *bracket* $(del(S, [i,\ 2n-2])\cup del(S',\ 2n-2),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ \sum_{i\in del(S, 2n-2)} R_2^{2n-2}\ R^i_{2n-i-1}\ R_1^{2n-1}$ *bracket* $(del(S, [i,\ 2n-2])\cup del(S',\ 2n-1),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ \sum_{i\in del(S, 2n-2)} R_2^{2n-2}\ R_1^{2n-2}\ R^i_{2n-i}$ *bracket* $(del(S, [i,\ 2n-2])\cup del(S',\ 2n-2),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ R_2^{2n-2}\ R_1^{2n-2}\ R_1^{2n-1}$ *bracket* $(del(S,\ 2n-2)\cup del(S', [2n-1,\ 2n-2]),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ R_2^{2n-2}\ R_1^{2n-2}\ S_{2n}$ *bracket* $(del(S,\ 2n-2)\cup del(S',\ 2n-2),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ R_2^{2n-2}\ S_{2n-1}\ R_2^{2n-2}$ *bracket* $(del(S,\ 2n-2)\cup del(S',\ 2n-2),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$+ R_2^{2n-2}\ S_{2n-1}\ R_1^{2n-1}$ *bracket* $(del(S,\ 2n-2)\cup del(S',\ 2n-1),$

$del(N, [2n-1,\ 2n])\cup del(N',\ 2n))$

$=$ *bracket* $(S, N)\ R_1^{2n-1}$ *bracket* $(del(S',\ 2n-1), del(N',\ 2n))$

$+$ *bracket* $(S, N)\ R_2^{2n-2}$ *bracket* $(del(S',\ 2n-2), del(N',\ 2n))$

$=$ *bracket* (S, N) *bracket* (S', N').

Then we have the results.

THEOREM **3.15.** *Let* $S=[i_0, i_0, \cdots, i_{n-1}]$ *and* $N=[j_0, j_1, \cdots, j_n]$ *satisfying that* $n\geqq 1$, $i_0=0$, $i_{n-1}=2n-2$, $i_{k-1}<i_k\leqq 2k$ *for each* $1\leqq k\leqq n-1$, $S\cap N=[\,]$ *and* $S\cup N=[0, 1, 2, \cdots, 2n]$. *And let* $S'=[i'_0, i'_1, \cdots, i'_m]$ *and* $N'=[j'_0, j'_1, \cdots, j'_m]$ *satisfying that* $m\geqq 2$, $i'_0=2n-2$, $i'_1=2n-1$, $i'_2=2n$, $i'_{k-1}<i'_k\leqq i'_0+2k-1$ *for each* $1\leqq k\leqq m$, $S'\cap N'=[\,]$ *and* $S'\cup N'=[2n-2,\ 2n-1,\ 2n, \cdots,\ 2n+2m-1]$. *Then we have that*

$d\,\{R_2^{2n-2}$ *bracket* $(del(S,\ 2n-2)\cup del(S',\ 2n-1), del(N, [2n-1,\ 2n])\cup N')$

$+ R_1^{2n-2}$ *bracket* $(del(S,\ 2n-2)\cup del(S',\ 2n), del(N, [2n-1,\ 2n])\cup N')\}$

$=$ *bracket* (S, N) *bracket* (S', N').

PROOF. By Corollary 2.2, we have that

$$d\{R_2^{2n-2}\ bracket\,(del\,(S,\ 2n-2)\cup del\,(S',\ 2n-1),\ del\,(N,[2n-1,\ 2n])\cup N')$$

$$+R_1^{2n-2}\ bracket\,(del\,(S,\ 2n-2)\cup del\,(S',\ 2n),\ del\,(N,[2n-1,\ 2n])\cup N')\}$$

$$=R_1^{2n-2}\ R_1^{2n-1}\ bracket\,(del\,(S,\ 2n-2)\cup del\,(S',\ 2n-1),\ del\,(N,[2n-1,\ 2n])\cup N')$$

$$+R_2^{2n-2}\ bracket\,(del\,(S,\ 2n-2)\cup S',\ del\,(N,\ 2n)\cup N')$$

$$+R_1^{2n-2}\ bracket\,(del\,(S,\ 2n-2)\cup S',\ del\,(N,\ 2n-1)\cup N')$$

By Lemma 1.2, we have that

$$=\mathrm{R}_1^{2n-2}\ \mathrm{R}_1^{2n-1}\ \mathrm{bracket}\,(\mathrm{del}\,(S,\ 2n-2)\cup \mathrm{del}\,(S',\ 2n-1),\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\mathrm{R}_2^{2n-2}\ \mathrm{R}_1^{2n-2}\ \mathrm{bracket}\,(\mathrm{del}\,(S,\ 2n-2)\cup \mathrm{del}\,(S',\ 2n-2),\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\sum_{i\in \mathrm{del}(S,\,2n-2)}\mathrm{R}_2^{2n-2}\ \mathrm{R}^{i}_{2n-i-1}\ \mathrm{bracket}\,(S,[i,\ 2n-2])\cup S',\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\mathrm{R}_2^{2n-2}\ \mathrm{R}_1^{2n-2}\ \mathrm{bracket}\,(\mathrm{del}\,(S,\ 2n-2)\cup \mathrm{del}\,(S',\ 2n-2),\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\mathrm{R}_1^{2n-2}\ \mathrm{R}_1^{2n-1}\ \mathrm{bracket}\,(\mathrm{del}\,(S,\ 2n-2)\cup \mathrm{del}\,(S',\ 2n-1),\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\sum_{i\in \mathrm{del}(S,\,2n-2)}\mathrm{R}_1^{2n-2}\ \mathrm{R}^{i}_{2n-i}\ \mathrm{bracket}\,(\mathrm{del}\,(S,[i,\ 2n-2])\cup S',\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\mathrm{R}_2^{2n-2}\ \mathrm{S}_{2n-1}\ \mathrm{bracket}\,(\mathrm{del}\,(S,\ 2n-2)\cup S',\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$+\mathrm{R}_1^{2n-2}\ \mathrm{S}_{2n}\ \mathrm{bracket}\,(\mathrm{del}\,(S,\ 2n-2)\cup S',\ \mathrm{del}\,(N,[2n-1,\ 2n])\cup N')$$

$$=\mathrm{bracket}\,(S,\ N)\ \mathrm{bracket}\,(S',\ N')$$

Then we have the results.

THEOREM 3.16. *Let* $S=[i_0,\ i_1,\cdots,\ i_{n-1}]$ *and* $N=[j_0,\ j_1,\cdots,\ j_n]$ *satisfying that* $n\geqq 2$, $i_0=0$, $i_{n-1}<2n-2$, $i_{k-1}<i_k\leqq 2k$ *for each* $1\leqq k\leqq n-1$, $S\cap N=[\ ]$ *and* $S\cup N=[0,1,2,\cdots,2n]$. *And let* $S'=[i'_0,\ i'_1,\cdots,\ i'_m]$ *and* $N'=[j'_0,\ j'_1,\cdots,\ j'_m]$ *satisfying that* $m\geqq 1$, $i'_0=2n-2$, $i'_1=2n-1$, $i'_{k-1}<i'_k\leqq i'_0+2k-1$ *for each* $1\leqq k\leqq m$, $S'\cap N'=[\ ]$, $S'\cup N'=[2n-2,\ 2n-1,\ 2n,\cdots,\ 2n+2m-1]$ *and if* $m\geqq 2$ *then* $i'_2>2n$. *Then we have that*

$$d\{R_2^{2n-2}\ bracket\,(S\cup del\,(S',[2n-2,\ 2n-1]),\ del\,(N,\ 2n-1)\cup del\,(N',\ 2n))$$

$$+R_1^{2n-1}\ bracket\,(S\cup del\,(S',[2n-2,\ 2n-1]),\ del\,(N,\ 2n-2)\cup del\,(N',\ 2n))\}$$

$$=bracket\,(S,\ N)\ bracket\,(S',\ N').$$

PROOF. By Corollary 2.2, we have that

$$d\{R_2^{2n-2}\ bracket\,(S\cup del\,(S',[2n-2,\ 2n-1]),\ del\,(N,\ 2n-1)\cup del\,(N',\ 2n))$$

$$+R_1^{2n-1}\ bracket\,(S\cup del\,(S',[2n-2,\ 2n-1]),\ del\,(N,\ 2n-2)\cup del\,(N',\ 2n))\}$$

$$=R_1^{2n-2}\ R_1^{2n-1}\ bracket\,(S\cup del\,(S',[2n-2,\ 2n-1]),\ del\,(N,\ 2n-1)\cup del\,(N',\ 2n))$$

$$+ R_2^{2n-2} \ \mathit{bracket}\,(S \cup \mathit{del}\,(S',\ 2n-2),\ N \cup \mathit{del}\,(N',\ 2n))$$

$$+ R_1^{2n-1} \ \mathit{bracket}\,(S \cup \mathit{del}\,(S',\ 2n-1),\ N \cup \mathit{del}\,(N',\ 2n))$$

By Lemma 1.2, we have that

$$= \sum_{i \in S} R_1^{2n-2} \ R_1^{2n-1} \ R^i_{2n-i} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S', [2n-2,\ 2n-1]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ R_1^{2n-2} \ R_1^{2n-1} \ S_{2n} \ \mathit{bracket}\,(S \cup \mathit{del}\,(S', [2n-2,\ 2n-1]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{[i,j] \subset S} R_2^{2n-2} \ R^i_{2n-i-1} \ R^j_{2n-j} \ \mathit{bracket}\,(\mathit{del}\,(S, [i, j]) \cup \mathit{del}\,(S',\ 2n-2),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_2^{2n-2} \ R^i_{2n-i-1} \ R_1^{2n-1} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S', [2n-2,\ 2n-1]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_2^{2n-2} \ R^i_{2n-i-1} \ S_{2n} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S',\ 2n-2),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_2^{2n-2} \ S_{2n-1} \ R^i_{2n-i} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S',\ 2n-2),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ R_2^{2n-2} \ S_{2n-1} \ R_1^{2n-1} \ \mathit{bracket}\,(S \cup \mathit{del}\,(S', [2n-2,\ 2n-1]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{[i,j] \subset S} R_1^{2n-1} \ R^i_{2n-i-1} \ R^j_{2n-j} \ \mathit{bracket}\,(\mathit{del}\,(S, [i, j]) \cup \mathit{del}\,(S',\ 2n-1),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_1^{2n-1} \ R^i_{2n-i-1} \ R_2^{2n-2} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S', [2n-1,\ 2n]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_1^{2n-1} \ R^i_{2n-i-1} \ S_{2n} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S',\ 2n-1),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_1^{2n-1} \ S_{2n-1} \ R^i_{2n-i} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S',\ 2n-1),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ R_1^{2n-1} \ S_{2n-1} \ R_2^{2n-2} \ \mathit{bracket}\,(S \cup \mathit{del}\,(S', [2n-2,\ 2n-1]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ \sum_{i \in S} R_1^{2n-1} \ R_1^{2n-2} \ R^i_{2n-i} \ \mathit{bracket}\,(\mathit{del}\,(S, i) \cup \mathit{del}\,(S', [2n-2,\ 2n-1]),$$
$$\mathit{del}\,(N, [2n-1,\ 2n]) \cup \mathit{del}\,(N',\ 2n))$$

$$+ R_1^{2n-1} \ R_1^{2n-2} \ S_{2n} \ \mathit{bracket}\,(S \cup \mathit{del}\,(S', [2n-2, 2n-1]),$$

$$\mathit{del}\,(N, [2n-1, 2n]) \cup \mathit{del}\,(S', 2n))$$

$$= \mathit{bracket}\,(S, N)\ \mathit{bracket}\,(S', N')$$

Then we have the results.

Finally we shall discuss the defining relations of the form $h_i(S)\,h_j(S') = \cdots$.

THEOREM **3.17.** *Let* $S_n = [i_0, i_1, \cdots, i_n]$, $N_n = [j_0, j_1, \cdots, j_n]$, $S' = [i'_0, i'_1, \cdots, i'_m]$ *and* $N' = [j'_0, j'_1, \cdots, j'_m]$ *satisfying that* $i \geqq 0$, $n \geqq 0$, $m \geqq 0$, $i_0 = i$, $i'_0 = i + 2n + 1$, $i_{k-1} < i_k \leqq i_0 + 2k - 1$ *for each* $1 \leqq k \leqq n$, $i'_{k-1} < i'_k \leqq i'_0 + 2k - 1$ *for each* $1 \leqq k \leqq m$, $S_n \cap N_n = [\,]$, $S' \cap N' = [\,]$, $S_n \cup N_n = [i, i+1, i+2, \cdots, i+2n+1]$ *and* $S' \cup N' = [i+2n+1, i+2n+2, i+2n+3, \cdots, i+2n+2m+3]$. *Then we have that*

$$d\ \mathit{bracket}\,(S_n \cup \mathit{del}\,(S', i+2n+1), \mathit{del}\,(N_n, i+2n+1) \cup N')$$

$$= \mathit{bracket}\,(S_n, N_n)\ \mathit{bracket}\,(S', N').$$

PROOF. By Corollary 2.2, we have that

$$d\ \mathit{bracket}\,(S_n \cup \mathit{del}\,(S', i+2n+1), \mathit{del}\,(N_n, i+2n+1) \cup N')$$

$$= \mathit{bracket}\,([i+2n+1 \mid S_n \cup \mathit{del}(S', i+2n+1)], [i+2n+1 \mid \mathit{del}(N_n, i+2n+1) \cup N'])$$

$$= \mathit{bracket}\,(S_n, N_n)\ \mathit{bracket}\,(S', N').$$

REMARK. This Theorem means that $h_i(n_1, n_2, \cdots, n_k)\ h_{i+2k+1}(m_1, m_2, \cdots, m_p) = 0$ for each $i \geqq 0$, $k \geqq 0$ and $p \geqq 0$.

THEOREM **3.18.** *Let* $S = [i_1, i_2, \cdots, i_n]$ *and* $N = [j_1, j_2, \cdots, j_{n+2}]$ *satisfying that* $i \geqq 0$, $n \geqq 0$, $i_1 \geqq i+2$, $i_{k-1} < i_k \leqq i+2k+1$ *for each* $1 \leqq k \leqq n$, $S \cap N = [\,]$ *and* $S \cup N = [i+2, i+3, i+4, \cdots, i+2n+3]$. *Then we have that*

$$d\ \mathit{bracket}\,([i, i \mid S], N)$$

$$= R_1^i\ \ \mathit{bracket}\,([i, i+1 \mid S], N) + \sum_{\alpha \in N} (R_{\alpha-i}^i)^2\ \ \mathit{bracket}\,([\alpha \mid S], \mathit{del}\,(N, \alpha)).$$

If there exist an index q *such that* $i_q = i + 2q + 1$ *then the summation is restricted as* $\sum_{\substack{\alpha \in N \\ i < \alpha < i+2p+1}}$, *where* p *is the least index* q *such that* $i_q = i + 2q + 1$.

PROOF. By Corollary 2.3, we have that

$$d\ \mathit{bracket}\,([i, i \mid S], N)$$

$$= \sum_{[\alpha, \beta, \gamma] \subset N} R_{\alpha-i}^i\ R_{\beta-i}^i\ R_{\gamma-i}^i\ \ \mathit{bracket}\,(S, [i \mid \mathit{del}\,(N, [\alpha, \beta, \gamma])])$$

$$+ \sum_{\alpha \in N} R_{\alpha-i-1}^{i+1}\ \ \mathit{bracket}\,([i, i \mid S], \mathit{del}\,(N, \alpha))$$

$$+\sum_{\alpha\in N}\sum_{\beta\in N} R^{\alpha}_{\beta-\alpha}\ (R^{i}_{\alpha-i})^2\ \textit{bracket}\ (S, \textit{del}\ (N, [\alpha,\beta]))$$

Since $i_k > i$ for each k, *bracket* $(S, [i \mid del\ (N, [\alpha, \beta, \gamma])]) = 0$. By Lemma 1.2, we have that

$$= R^{i+1}_{1}\ \sum_{\alpha\in N} R^{i+1}_{\alpha-i-1}\ \textit{bracket}\ ([i \mid S], \textit{del}\ (N,\alpha))$$

$$+\sum_{\alpha\in N} (R^{i}_{\alpha-i})^2\ \textit{bracket}\ ([\alpha \mid S], \textit{del}\ (N,\alpha))$$

$$= R^{i+1}_{1}\ \textit{bracket}\ ([i, i+1 \mid S], N) + \sum_{\alpha\in N} (R^{i}_{\alpha-i})^2\ \textit{bracket}\ ([\alpha \mid S], \textit{del}\ (N,\alpha))$$

We assume that there exist an index q such that $i_q = i+2q+1$ and p is the least such index. If $\alpha \geqq i+2p+1$ then $[\alpha \mid S]$ contains $p-1$ items less than $i+2p+1$ and *del* (N, α) contains p items less than $i+2p+1$. So we have that bracket $([\alpha \mid S], del\ (N, \alpha)) = 0$ in such a case. Then we have the results.

REMARK. This is proved in [6 : Theorem 4.2].

THEOREM **3.19.** *Let* $S = [i_0, i_1, \cdots, i_{n+1}]$ *and* $N = [j_1, j_2, \cdots, j_n]$ *satisfying that* $n \geqq 0$, $i \geqq 0$, $i_0 = i$, $i_{k-1} < i_k \leqq i+2k-1$ *for each* $1 \leqq k \leqq n+1$, $S \cap N = [\,]$ *and* $S \cup N = [i, i+1, i+2, \cdots, i+2n+1]$. *Then we have that*

$$d\ \textit{bracket}\ (S, [i+2n+3, i+2n+3 \mid N])$$

$$= R^{i+2n+2}_{1}\ \textit{bracket}\ (S, [i+2n+2, i+2n+3 \mid N])$$

$$+\sum_{s\leq p\leq n+1} (R^{i_p}_{i+2n-i_p+3})^2\ \textit{bracket}\ (\textit{del}\ (S, i_p), [i_p \mid N]),$$

where s is the largest index q such that $i_q = i+2q-1$ $(1 \leqq q \leqq n+1)$.

PROOF. By Corollary 2.4, we have that

$$d\ \textit{bracket}\ (S, [i+2n+3, i+2n+3 \mid N])$$

$$= \sum_{\alpha\in S} (R^{\alpha}_{i+2n-\alpha+3})^2\ \sum_{k\in S} R^{k}_{\alpha-k}\ \textit{bracket}\ (\textit{del}\ (S, [k,\alpha]), N)$$

$$+ R^{i+2n+2}_{1}\ \textit{bracket}\ (S, [i+2n+2, i+2n+3 \mid N])$$

$$= R^{i+2n+2}_{1}\ \textit{bracket}\ (S, [i+2n+2, i+2n+3 \mid N])$$

$$+\sum_{\alpha\in S} (R^{\alpha}_{i+2n-\alpha+3})^2\ \textit{bracket}\ (\textit{del}\ (S,\alpha), [\alpha \mid N])$$

If $p < s$ then *del* (S, i_p) contains $s-1$ items less than $i+2s-1$ and $[i_p \mid N]$ contains items less than $i+2s-1$. So in this case *bracket* $(del\ (S, i_p), [i_p \mid N]) = 0$. Then we have the results.

REMARK. This is proved in [7 : Theorem 2].

The proofs of the following Theorems are essentially same. So we omit these proofs.

THEOREM **3.20.** *Let* $S_n = [i_1, i_2, \cdots, i_n]$, $N_n = [j_1, j_2, \cdots, j_n]$, $S' = [i'_1, i'_2, \cdots, i'_m]$ *and* $N' = [j'_1, j'_2, \cdots, j'_m]$ *satisfying that* $n \geqq 2$, $m \geqq 2$, $i \geqq 0$, $i_1 = i$, $i_n = i+2n-3$, $i'_1 = i+2n-2$,

$i'_2=i+2n-1$, $i_{k-1}<i_k\leqq i+2k-3$ *for each* $1\leqq k\leqq n$, $i'_{k-1}<i'_k\leqq i'_1+2k-3$ *for each* $1\leqq k\leqq m$, $S_n\cap N_n=[\,]$, $S_n\cup N_n=[i, i+1, i+2,\cdots, i+2n-1]$, $S'\cap N'=[\,]$ *and* $S'\cup N'=[i+2n-2, i+2n-1, i+2n,\cdots, i+2n+2m-3]$. *Then we have that*

$$d\{R_1^{i+2n-3}\ bracket\ (del\ (S_n, i+2n-3)\cup del\ (S', i+2n-1),$$
$$del\ (N_n, [i+2n-2, i+2n-1]\cup N')$$
$$+R_2^{i+2n-3}\ bracket\ (del\ (S_n, i+2n-3)\cup del\ (S', i+2n-2),$$
$$del\ (N_n, [i+2n-2, i+2n-1])\cup N')$$
$$+bracket\ (del\ (S_n, i+2n-3), [i+2n-3\mid del\ (N_n, [i+2n-2, i+2n-1])])$$
$$\times bracket\ ([i+2n-3, i+2n-3\mid del\ (S', [i+2n-2, i+2n-1])], N')\}$$
$$=bracket\ (S_n, N_n)\ bracket\ (S', N')$$
$$+bracket\ (del\ (S_n, i_n), [i_n\mid del\ (N_n, [j_{n-1}, j_n])])$$
$$\times\{\sum_{1\leq p\leq m}(R_{j'_p-i_n}^{i_n})^2\ bracket\ ([j'_p\mid del\ (S', [i'_1, i'_2])], del\ (N', j'_p))\}$$

THEOREM **3.21.** *Let* $S_n=[i_1, i_2,\cdots, i_n]$ *and* $N_n=[j_1, j_2,\cdots, j_n]$, $S'=[i'_1, i'_2,\cdots, i'_m]$ *and* $N'=[j'_1, j'_2,\cdots, j'_m]$ *satisfying that* $n\geqq 2$, $m\geqq 2$, $i\geqq 0$, $i_1=i$, $i_n=i+2n-3$, $i'_1=i+2n-3$, $i'_2=i+2n-2$, $i'_3=i+2n$ *if* $m\geqq 3$, $i_{k-1}<i_k\leqq i_1+2k-3$ *for each* $2\leqq k\leqq n$, $i'_{k-1}<i'_k\leqq i'_1+2k-3$ *for each* $2\leqq k\leqq m$, $S_n\cap N_n=[\,]$, $S_n\cup N_n=[i, i+1, i+2,\cdots, i+2n-1]$, $S'\cap N'=[\,]$ *and* $S'\cup N'=[i+2n-3, i+2n-2, i+2n-1,\cdots, i+2n+2m-4]$. *Then we have that*

$$d\ bracket\ (S_n\cup del\ (S', i+2n-2), del\ (N_n, i+2n-2)\cup N')$$
$$=bracket\ (S_n, N_n)\ bracket\ (S', N').$$

THEOREM **3.22.** *Let* $S_n=[i_1, i_2,\cdots, i_n]$, $N_n=[j_1, j_2,\cdots, j_n]$, $S'=[i'_1, i'_2,\cdots, i'_m]$ *and* $N'=[j'_1, j'_2,\cdots, j'_m]$ *satisfying that* $n\geqq 2$, $m\geqq 3$, $i\geqq 0$, $i_1=i$, $i_n=i+2n-3$, $i'_1=i+2n-3$, $i'_2=i+2n-2$, $i'_3=i+2n-1$, $i_{k-1}<i_k\leqq i+2k-3$ *for each* $2\leqq k\leqq n$, $i'_{k-1}<i'_k\leqq i'_1+2k-3$ *for each* $2\leqq k\leqq m$, $S_n\cap N_n=[\,]$, $S_n\cup N_n=[i, i+1, i+2,\cdots, i+2n-1]$, $S'\cap N'=[\,]$ *and* $S'\cup N'=[i+2n-3, i+2n-2, i+2n-1,\cdots, i+2n+2m-4]$. *Then we have that*

$$d\{R_1^{i+2n-3}\ bracket\ (S_n\cup del\ (S', [i'_1, i'_3]), del\ (N_n, [j_{n-1}, j_n])\cup N')$$
$$+R_2^{i+2n-3}\ bracket\ (S_n\cup del\ (S', [i'_1, i'_2]), del\ (N_n, [j_{n-1}, j_n])\cup N')\}$$
$$=bracket\ (S_n, N_n)\ bracket\ (S', N').$$

THEOREM **3.23.** *Let* $S=[i_1, i_2,\cdots, i_n]$ *and* $N=[j_1, j_2,\cdots, j_n]$ *satisfying that* $n\geqq 3$, $i\geqq 0$, $i_1=i$, $i_2=i+1$, $i_3=i+3$, $i_{k-1}<i_k\leqq i_1+2k-3$ *for each* $2\leqq k\leqq n$, $S\cap N=[\,]$ *and* $S\cup N=[i, i+1, i+2,\cdots, i+2n-1]$. *Then we have that*

$$d\ bracket\ ([i,\ i+1\ |\ del\ (S,\ i+3)],\ [i+2\ |\ N])$$

$$=bracket\ ([i,\ i+1],\ [i+2,\ i+3])\ bracket\ (S,\ N)$$

$$+\sum_{\alpha\in del(N,i+2)}\{(R^{i}_{\alpha-i})^2\ (R^{i+1}_{1})^2+(R^{i}_{2})^2\ (R^{i+1}_{\alpha-i-1})^2\ \}$$

$$\times\ bracket\ (del\ ([\alpha\ |\ S],\ [i,\ i+1,\ i+3]),\ del\ (N,\ [i+2,\ \alpha])).$$

THEOREM **3.24.** *Let* $S=[i_1,\ i_2,\cdots,\ i_n]$ *and* $N=[j_1,\ j_2,\cdots,\ j_n]$ *satisfying that* $i\geqq 0$, $n\geqq 3$, $i_1=i$, $i_2=i+1$, $i_3=i+2$, $i_4\geqq i+4$ *if* $n\geqq 4$, $i_{k-1}<i_k\leqq i_1+2k-3$ *for each* $2\leqq k\leqq n$, $S\cap N=[\,]$ *and* $S\cup N=[i,\ i+1,\ i+2,\cdots,\ i+2n-1]$. *Then we have that*

$$d\ bracket\ (del\ ([i,\ i+1\ |\ S],\ i+2),\ [i+3\ |\ N])$$

$$=bracket\ ([i,\ i+1],\ [i+2,\ i+3])\ bracket\ (S,\ N)$$

$$+\sum_{\alpha\in del(N,i+3)}\{(R^{i}_{\alpha-i})^2\ (R^{i+1}_{2})^2+(R^{i}_{3})^2\ (R^{i+1}_{\alpha-i-1})^2\ \}$$

$$\times\ bracket\ (del\ ([\alpha\ |\ S],\ [i,\ i+1,\ i+2]),\ del\ (N,\ [i+3,\ \alpha])).$$

THEOREM **3.25.** *Let* $S=[i_1,\ i_2,\cdots,\ i_n]$ *and* $N=[j_1,\ j_2,\cdots,\ j_n]$ *satisfying that* $i\geqq 0$, $n\geqq 4$, $i_1=i$, $i_2=i+1$, $i_3=i+2$, $i_4=i+3$, $i_{k-1}<i_k\leqq i_1+2k-3$ *for each* $2\leqq k\leqq n$, $S\cap N=[\,]$ *and* $S\cup N=[i,\ i+1,\ i+2,\ \cdots,\ i+2n-1]$. *Then we have that*

$$d\ \{R^{i+1}_{1}\ \ bracket\ ([i\ |\ del\ (S,\ i+3)],\ N)$$

$$+R^{i+1}_{2}\ \ bracket\ ([i\ |\ del\ (S,\ i+2)],\ N)$$

$$+\sum_{\alpha\in N}(R^{i}_{\alpha-i})^2\ \ bracket\ ([i+1,\ \alpha\ |\ S],\ [i,\ i+2,\ i+3]),\ del\ (N,\ \alpha))\}$$

$$=bracket\ ([i,\ i+1],\ [i+2,\ i+3])\ bracket\ (S,\ N)$$

$$+\sum_{\alpha\in N}\ \sum_{\beta\in del(N,\alpha)}(R^{i}_{\alpha-i})^2\ (R^{i+1}_{\beta-i-1})^2\ \ bracket\ ([\alpha,\ \beta\ |\ del\ (S,\ [i,\ i+1,\ i+2,\ i+3])],$$

$$del\ (N,\ [\alpha,\ \beta])).$$

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