On some inequalities

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Mr. Pocklington proved the following two theorems.

Theorem A.

Suppose that

$$\sum_{\nu=1}^{n} x_\nu y_\nu \geqq a^p, \quad \sum_{\nu=1}^{n} x_\nu \leqq ka$$

then

$$\sum_{\nu=1}^{s} x_\nu \geqq a \quad \text{or} \quad \sum_{\nu=1}^{s} y_\nu \geqq a^{p-1}$$

for $s \geqq k$

Theorem B.

Suppose that

$$\sum_{\nu=1}^{n} x_\nu \rho \geqq a^p \quad \text{with} \quad p \geqq 2$$

and

$$\sum_{\nu=1}^{n} x_\nu \leqq ka$$

then

$$\sum_{\nu=1}^{s} x_\nu \geqq a \quad \text{for} \quad s \geqq k$$

where a, k and p are positive and $\{x_n\}$, $\{y_n\}$ are positive non-increasing.

Mr. Nomura extended Pocklington's theorems.

The object of the present paper is to prove the following inequalities.

We suppose that m is a positive integer and $\{x_n\}$ and $\{y_n\}$ are positive non-increasing sequences.

Theorem 1.

Suppose that

(1) $\sum_{\nu=1}^{n} x_\nu^m \leqq ka$

(2) $\sum_{\nu=1}^{s} x_\nu^m \leqq a$

(3) $\sum_{\nu=1}^{s'} y_\nu^m \leqq b$, for $n \geqq s \geqq s' \geqq k$

then

$$\sum_{\nu=1}^{n} x_\nu^m y_\nu^m \leqq ab$$

To prove the above theorem, we get

$$\sum_{\nu=1}^{s} x_\nu^m = X_s, \quad \sum_{\nu=1}^{n} x_\nu^m = X_n,$$

then

$$\sum_{\nu=1}^{n} x_\nu{}^m y_\nu{}^m \leqq y_1{}^m X_s + y_s{}^m (X_n - X_s)$$
$$= (y_1{}^m - y_s{}^m) X_s + y_s{}^m X_n$$
$$\leqq (y_1{}^m - y_s{}^m) a + y_s{}^m ka$$
$$\leqq \{y_1{}^m + y_s{}^m (k-1)\} a$$
$$\leqq \{y_1{}^m + y^m{}_{s'} (s'-1)\} a$$
$$\leqq (y_1{}^m + y_2{}^m + \cdots + y^m{}_{s'}) a$$
$$\leqq ab$$

We get the following corollary easily.

Corollary.

Suppose

$$\sum_{\nu=1}^{n} x_\nu{}^m y_\nu{}^m \geqq ab \quad \text{and} \quad \sum_{\nu=1}^{n} x_\nu{}^m \leqq ka, \quad \text{then we get}$$

$$\sum_{t=1}^{s} x_\nu{}^m \geqq a \quad \text{or} \quad \sum_{\nu=1}^{s} y_\nu{}^m \geqq b \quad \text{for} \quad s \geqq k$$

Theorem 2.

Suppose that

$$\sum_{\nu=1}^{n} x_\nu{}^m \leqq ka, \quad \sum_{\nu=1}^{s} x_\nu{}^m \leqq a, \quad \sum_{\nu=1}^{s'} y_\nu{}^m \leqq b \quad \text{for} \quad n \geqq s \geqq s' \geqq k$$

and

(4) $$\frac{\varphi(x_\nu)}{x_\nu{}^m} \leqq \frac{A}{a}$$

(5) $$\frac{\varphi(y_\nu)}{y_\nu{}^m} \leqq \frac{B}{b}$$

then

$$\Sigma \varphi(x_\nu) \cdot \varphi(y_\nu) \leqq AB$$

This theorem will be proved as follows.

$$\Sigma \varphi(x_\nu) \cdot \phi(y_\nu) \leqq \Sigma \frac{A}{a} \cdot \frac{B}{b} \cdot x_\nu{}^m y_\nu{}^m = \frac{AB}{ab} \Sigma x_\nu{}^m y_\nu{}^m \leqq \frac{AB}{ab} \cdot ab = AB$$

Corollary 1.

Suppose that

(1) $\dfrac{\varphi(t)}{t^m}$ is a non-decreasing function of t for $t_1 \geqq t_2 \geqq \cdots \geqq t_n > 0$

(2) $\displaystyle\sum_{\nu=1}^{n} t_\nu{}^m \leqq ka, \quad \sum_{\nu=1}^{s} t_\nu{}^m \leqq a$ for $s \geqq k$

then

$$\sum_{\nu=1}^{n} t_\nu{}^m \phi(t_\nu) \leqq a \phi(a)$$

To prove the above corollary, we put

$$x_\nu = y_\nu = t_\nu, \quad a = b, \quad A = a, \quad B = \varphi(a)$$

then

$$\frac{\psi(a)}{a} \geqq \frac{\psi(t_1)}{t_1^m} \geqq \frac{\psi(t_2)}{t_2^m} \geqq \cdots \geqq \frac{\psi(t_s)}{t_s^m} \geqq \cdots \geqq \frac{\varphi(t_n)}{t_n^m},$$

$$\frac{\varphi(x_\nu)}{x_\nu^m} = \frac{\varphi(t_\nu)}{t_\nu^m} = 1, \quad \frac{\varphi(y_\nu)}{y_\nu^m} = \frac{\varphi(t_\nu)}{t_\nu^m} \leqq \frac{\varphi(a)}{a}$$

By the theorem 2.

$$\Sigma\varphi(x_\nu)\psi(y_\nu) = \Sigma\varphi(t_\nu)\psi(t_\nu) = \Sigma t_\nu^m \cdot \psi(t_\nu) \leqq 1 \cdot \frac{\psi(a)}{a} = a\psi(a)$$

Corollary 2.

Suppose that

(1) $\frac{\psi(t)}{t^m}$ is a non-decreasing function of t for $t_1 \geqq t_2 \geqq \cdots \geqq t_n > 0$

(2) $\sum_{\nu=1}^{n} t_\nu^m \psi(t_\nu) \geqq a\psi(a)$

(3) $\sum_{\nu=1}^{n} t_\nu^m \leqq ka$

then

$$\sum_{\nu=1}^{s} t_\nu^m \geqq a \quad \text{for} \quad s \geqq k$$

Theorem 3.

Suppose that

1° $\sum_{\nu=1}^{n} x_\nu^m \leqq ka, \quad \sum_{\nu=1}^{s} x_\nu^m \leqq a$

for $n \geqq s \geqq s' \geqq k$

2° $\sum_{\nu=1}^{s_{i-1}} y_{i\nu}^m \leqq k_i a_i, \quad \sum_{\nu=1}^{s_i} y_{i\nu}^m \leqq a_i$

for $s'_{i-1} \geqq s_i \geqq s_i' \geqq k_i$ $(i = 1, 2, \cdots\cdots, l-1)$, $s_0' = s'$

3° $\sum_{\nu=1}^{s'_{l-1}} y_{l\nu}^m \leqq a_l$

for $y_{i1} \geqq y_{i2} \geqq \cdots \geqq y_{in}$ $(i = 1, 2, \cdots\cdots, l)$

then

$$\sum_{\nu=1}^{n} x_\nu^m y_{1\nu}^m y_{2\nu}^m \cdots y_{l\nu}^m \leqq a a_1 a_2 \cdots a_l$$

To prove the above theorem we put

$$z_\nu = y_{1\nu} y_{2\nu} \cdots y_{l\nu}$$

Hence

$$z_1 \geqq z_2 \geqq \cdots \geqq z_n > 0$$

Therefore we get

$$\sum_{\nu=1}^{n} x_\nu^m z_\nu^m \leqq a \sum_{\nu=1}^{s'} z_\nu^m = a \sum_{\nu=1}^{s'} y_{1\nu}^m y_{2\nu}^m \cdots y_{l\nu}^m \tag{1}$$

We put

$$w_\nu = y_{2\nu} y_{3\nu} \cdots y_{l\nu}$$

then

$w_1 \geqq w_2 \geqq \cdots \geqq w_{s'} > 0$

therefore

$$\sum_{\nu=1}^{s'} z_\nu{}^m = \sum_{\nu=1}^{s'} y_{1\nu}{}^m w_\nu{}^m \leqq a_1 \sum_{\nu=1}^{s_1'} w_\nu{}^m \qquad (2)$$

From (1), (2), we get

$$\sum_{\nu=1}^{n} x_\nu{}^m z_\nu{}^m \leqq a a_1 \sum_{\nu=1}^{s_1'} w_{\nu m}$$

Similary we get

$$\sum_{\nu=1}^{n} x_\nu{}^m z_\nu{}^m \leqq a a_1 a_2 \cdots a_{l-1} \sum_{\nu=1}^{s'_{l-1}} y^m{}_{l\nu} \leqq a a_1 a_2 \cdots a_l$$

Therefore

$$\sum_{\nu=1}^{n} x_\nu{}^m y_{1\nu}{}^m y_{2\nu}{}^m \cdots y_{l\nu}{}^m \leqq a a_1 a_2 \cdots a_l$$

Theorem 4.

Suppose that

1° $\sum_{\nu=1}^{n} x_\nu{}^m \leqq ka,\ \sum_{\nu=1}^{s} x_\nu{}^m \leqq a$ for $n \geqq s \gtreqless s' \geqq k$

2° $\sum_{\nu=1}^{s'_{l-1}} y_{l\nu m} \leqq k_l a_l,\quad \sum_{\nu=1}^{s_l} y_{l\nu}{}^m \leqq a_l$ for $s_{l-1}' \geqq s_i \geqq s_{l-1}' \geqq k_i$ $(i=1, 2, \cdots, l-1)$, $s_0' = s'$

3° $\sum_{\nu=1}^{s'_{l-1}} y_{l\nu}{}^m \leqq a_l$

for $y_{i1} \geqq y_{i2} \geqq \cdots \geqq y_{in}$ $(i=1, 2, \cdots, l)$

4° $\dfrac{\varphi(x_\nu)}{x_\nu{}^m} \leqq \dfrac{A}{a},\quad \dfrac{\varphi_i(y_{i\nu})}{y_{i\nu}{}^m} \leqq \dfrac{A_i}{a_i}$ $(i=1, 2, \cdots, l)$

then

$$\sum_{\nu=1}^{n} \varphi(x_\nu)\varphi_1(y_{1\nu})\varphi_2(y_{2\nu})\cdots\varphi_l(y_{l\nu}) \leqq A A_1 \cdots A_l$$

We can easily prove as follows.

$$\sum_{\nu=1}^{n} \varphi(x_\nu)\varphi_1(y_{1\nu})\varphi_2(y_{2\nu})\cdots\varphi_l(y_{l\nu})$$

$$\leqq \Sigma \frac{A}{a} \cdot x_\nu{}^m \cdot \frac{A_1}{a_1} y_{1\nu}{}^m \cdot \frac{A_2}{a_2} y_{2\nu}{}^m \cdots \frac{A_l}{a_l} y_{l\nu}{}^m$$

$$\leqq \frac{A A_1 A_2 \cdots A_l}{a a_1 a_2 \cdots a_l} \Sigma x_\nu{}^m y_{1\nu}{}^m y_{2\nu}{}^m \cdots y_{l\nu}{}^m$$

$$= \frac{A A_1 A_2 \cdots A_l}{a a_1 a_2 \cdots a_l} a a_1 a_2 \cdots a_l$$

$$= A A_1 A_2 \cdots A_l$$

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