Theorems on Riesz's Method of Summation

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T. Kojima proved the following theorem.

Theorem A.

A. necessary and sufficient condition that the convergence of a sequence

$\left\{\frac{b_1\xi_1+b_2\xi_2+\cdots+b_n\xi_n}{b_1+b_2+\cdots+b_n}\right\}$ implies that of $\left\{\frac{c_1\xi_1+c_2\xi_2+\cdots+c_n\xi_n}{c_1+c_2+\cdots+c_n}\right\}$ are that

(1) the series $\sum_{n=1}^{\infty} c_n$ converges but not to zero or diverges

(2) $$\left\{\sum_{i=1}^{n-1}\left|\frac{b_i b_{i+1}-c_{i+1}c_i}{b_i b_{i+1}}(b_1+b_2+\cdots+b_i)\right.\right.$$

$$\left.\left.+\left|\frac{c_n}{b_n}(b_1+b_2+\cdots+b_n)\right|\right\}\div\left|\sum_{i=1}^{n}c_i\right|<M \qquad (n=1,2,\cdots\cdots)$$

where M is a fixed positive number. This theorem is rewritten as follows.

Theorem A′

A necessary and sufficient condition that $(R,q_n)\subset(R,p_n)$ is that

$$\frac{1}{P_n}\sum_{\nu=0}^{n-1}\left|\frac{p_\nu}{q_\nu}-\frac{p_{\nu+1}}{q_{\nu+1}}\right|Q_\nu+\frac{p_nQ_n}{q_nP_n}=O\ (1) \qquad (1)$$

where summation (R,p_n) means Riesz's mean, i.e.

$$\sigma_n=\frac{p_0s_0+p_1s_1+\cdots+p_ns_n}{P_n},$$

$$P_n=p_0+p_1+\cdots+p_n$$

tends to σ as $n\to\infty$.

Hence $(R,q_n)\subset(R,p_n)$ means that the series summable (R,q_n) is also summable (R,p_n), and their sum are equal.

Mr. S. Izumi and G. Hayashi proved the analogous theorem for Nörlund's method of summation which contains the consistency theorem for Cesaro's method of summation.

In section 1 of this paper, we will prove Kojima's theorem by using Izumi's method. In section 2 we will prove some theorems on Riesz's summation.

§ 1. We put $s_n=\sum_{\nu=0}^{n}a_\nu$ for a given series $\sum_{n=0}^{\infty}a_n$.

If the series $\sum_{n=0}^{\infty}a_n$ is summable (R,q), then

$$\rho_n=\frac{q_0s_0+q_1s_1+\cdots+q_ns_n}{Q_n} \qquad (1)$$

where $Q_n=q_0+q_1+\cdots+q_n$ tends to a limit ρ as $n\to\infty$.

From (1)

$$\left.\begin{array}{l} \rho_0 Q_n = q_0 s_0 \\ \rho_1 Q_1 = q_0 s_0 + q_1 s_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \rho_n Q_n = q_0 s_0 + q_1 s_1 + \cdots + q_n s_n \end{array}\right\} \qquad (2)$$

Without loss of generality we can put $q_0 = 1$.
If we solve the above equation (2), we get

$$S_n = \left|\begin{array}{ccccc} 1 & & & & \rho_0 Q_0 \\ 1 & q_1 & q_0 & & \rho_1 Q_1 \\ 1 & q_1 & q_2 & & \rho_2 Q_2 \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & q_1 & q_2 & & \rho_n Q_n \end{array}\right| \Bigg/ \left|\begin{array}{ccccc} 1 & & & & \\ 1 & q_1 & & & \\ 1 & q_1 & q_2 & & \\ \vdots & & & \ddots & \\ 1 & & & & q_n \end{array}\right|$$

$$= (-1)^n \left|\begin{array}{llllll} Q_0 \rho_0 & 1 & & & & \\ Q_1 \rho_1 & 1 & q_1 & & & \\ Q_2 \rho_2 & 1 & q_1 & q_2 & & \\ \cdots\cdots & \cdots & \cdots & \cdots & \cdots & \\ Q_{n-1} \rho_{n-1} & \cdots & \cdots & \cdots & q_{n-1} & \\ Q_n \rho_n & 1 & q_1 & q_2 & \cdots & q_n \end{array}\right| \Bigg/ (q_1 q_2 \cdots q_n)$$

$$= Q_0 \rho_0 \Delta_0 + Q_1 \rho_1 \Delta_1 + \cdots\cdots + Q_n \rho_n \Delta_n$$

where

$$\Delta_0 = \Delta_1 = \cdots\cdots = \Delta_{n-2} = 0,$$

$$\Delta_{n-1} = (-1)^n (-1)^{n-1} \left|\begin{array}{lllll} 1 & & & & \\ 1 & q_1 & & & \\ 1 & q_1 & q_2 & & \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & q_1 & q_2 & \cdots & q_{n-1} \end{array}\right| \Bigg/ (q_1 q_2 \cdots q_n)$$

$$= -\frac{q_1 q_2 \cdots\cdots q_{n-1}}{q_1 q_2 \cdots\cdots q_n} = -\frac{1}{q_n}$$

$$\Delta_n = (-1)^n (-1)^n \left|\begin{array}{lllll} 1 & & & & \\ 1 & q_1 & & & \\ 1 & q_1 & q_2 & & \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & q_1 & q_2 & \cdots\cdots & q_{n-1} \end{array}\right| \Bigg/ (q_1 q_2 \cdots q_n)$$

$$= \frac{1}{q_n}$$

Therefore

$$s_n = -Q_{n-1} \rho_{n-1} \frac{1}{q_n} + Q_n \rho_n \frac{1}{q_n}$$

$$\sigma_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu \tag{3}$$

$$= \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu\left(-\frac{Q_{\nu-1}\cdot\rho_{\nu-1}}{q_0}+\frac{Q_\nu\rho_\nu}{q_\nu}\right)$$

$$= \frac{1}{R_n}\sum_{\nu=0}^{n-1}\left(\frac{p_0}{q_\nu}-\frac{p_{\nu+1}}{q_{\nu+1}}\right)Q_\nu\rho_\nu+\frac{Q_np_n}{q_nP_n}\rho_n.$$

Therefore if the transformation matrix from the sequence $\{\rho_\nu\}$ to the sequence $\{\sigma_n\}$ is reqular, then from (R, q) summability of the series $\sum_{n=0}^{\infty} a_n$ we can conclude (R,p) sum-mability of this series.

§. 2.

From (3)

$$s_n-\sigma_n = s_n-\frac{1}{P_n}\sum_{\nu=0}^{n}p_\nu s_\nu = \frac{1}{P_n}\left\{P_ns_n-\sum_{\nu=0}^{n}\rho_\nu s_\nu\right\}$$

$$= \frac{1}{P_n}\left\{(p_0+p_1+\cdots+p_n)s_n-(p_0s_0+p_1s_1+\cdots+p_ns_n)\right\}$$

$$= \frac{1}{P_n}\left\{p_0(s_n-s_0)+p_1(s_n-s_1)+\cdots p_{n-1}(s_n-s_{n-1})\right\}$$

$$= \frac{1}{P_n}\left\{p_0(a_1+a_2+\cdots+a_n)+p_1(a_2+a_3+\cdots+a_n)+\cdots+p_{n-1}a_n\right\}$$

$$= \frac{1}{P_n}\left\{p_0a_1+(p_0+p_1)a_2+\cdots\cdots+(p_0+p_1+\cdots+p_{n-1})a_n\right\}$$

$$= \frac{1}{P_n}\sum_{\nu=1}^{n}P_{\nu-1}a_\nu$$

Thus we get the following theorem.

Theorem 1.

Suppose that the series $\sum_{n=0}^{\infty} a_n$ is summable (R,p_n) to the value s.

A necessary and sufficient condition that $\sum_{n=0}^{\infty} a_n = s$ is that

$$\sum_{\nu=1}^{n}P_{\nu-1}a_\nu = O(P_n)$$

where $P_n = p_0+p_1+\cdots+p_n$

Theorem 2.

If the series $\sum_{n=0}^{\infty} a_n$ is summable (R,p_n) and $P_{n-1}a_n = O(1)$, then Σa_n converges.

Proof.

By $P_{n-1}a_n=O(1)$, $|P_{n-1}a_n|<K$ for all n and $P_{n-1}a_n<\varepsilon$ fro $n>n_0$.

Therefore

$$\left|\frac{P_0a_1+P_1a_2+\cdots+P_{n-1}a_n}{P_n}\right| \leqq \left|\frac{P_0a_1+P_1a_2+\cdots+P_{n_0-1}a_{n_0}}{P_n}\right|$$

$$+\left|\frac{P_{n_0}a_{n_0+1}+\cdots+P_{n-1}a_n}{P_n}\right| \leqq \frac{n_0K}{P_n}+\frac{(n-n_0)\varepsilon}{P_n}$$

Take

$$P_n > \frac{n_0(K-\varepsilon)}{\varepsilon},$$

we get

$$\frac{n_0 K}{P_n} + \frac{(n-n_0)\varepsilon}{P_n} < 2\varepsilon$$

Hence

$$\left| \frac{P_0 a_1 + \cdots + P_{n-1} a_n}{P_n} \right| < 2\varepsilon$$

Therefore by the theorem 1, the series $\sum_{n=0}^{\infty} a_n$ converges.

Theoram 3.

If the series $\sum_{n=0}^{\infty} a_n$ is summable (R, P_n) to the value s, then $s_n = O\left(\frac{P_n}{p_n}\right) + s$

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