A fifth-order dispersive partial differential equation for curve flow on the sphere

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Abstract

The initial value problem for a fifth-order nonlinear dispersive partial differential equation describing the curve flow on the sphere is considered. A typical example of the equation arises in a hierarchy of completely integrable systems containing one-dimensional classical Heisenberg ferromagnetic spin model. This paper establishes the local existence and uniqueness of a solution to the initial value problem under the periodic boundary condition. The proof is based on the energy method combined with a kind of gauge transformation to overcome the difficulty of a loss of derivative.

Keywords: nonlinear dispersive partial differential equation, local existence and uniqueness, loss of derivative, energy method, gauge transformation
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1. Introduction

1.1. Setting of the problem and the background

Let $N \geqslant 1$ be an integer, X be either the real line $\mathbb{R}$ or the one-dimensional flat torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, and let $\mathbb{S}^N$ be the N-dimensional unit sphere in $\mathbb{R}^{N+1}$ centered at the origin. We consider the initial value problem for a fifth-order nonlinear partial

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differential equation (PDE) of the form

$$
\begin{aligned}
u_t = {} & b_1 u_{xxxxx} + 5b_1(u_{xxxx}, u_x)u + 10b_1(u_{xxx}, u_{xx})u + b_2|u_x|^2 u_{xxx} \\
& + b_3(u_{xxx}, u_x)u_x + b_4(u_{xx}, u_x)u_{xx} + b_5|u_{xx}|^2 u_x \\
& + b_6|u_x|^2(u_{xx}, u_x)u + b_7|u_x|^4 u_x, \qquad (1.1) \\
u(0, x) = {} & u_0(x), \qquad (1.2)
\end{aligned}
$$

where $u = u(t,x) : \mathbb{R} \times X \to \mathbb{S}^N \subset \mathbb{R}^{N+1}$ being a curve flow on $\mathbb{S}^N$ is the unknown function and $u_0 = u_0(x) : X \to \mathbb{S}^N$ is a given initial function. The standard inner product and the norm in $\mathbb{R}^{N+1}$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$ respectively, that is, $(u,v) = u_1v_1 + u_2v_2 + \cdots + u_{N+1}v_{N+1}$ and $|u| = \sqrt{(u,u)}$ for any $u = (u_1, u_2, \dots, u_{N+1}), v = (v_1, v_2, \dots, v_{N+1}) \in \mathbb{R}^{N+1}$. Throughout this paper, it is assumed that $b_1, b_2, \dots, b_7$ are real constants which satisfy $b_1 \neq 0$ and $-3b_2 - b_4 + b_6 = 0$.

The setting for $b_1, b_2, \dots, b_7$ in (1.1) comes from the requirement for the solution u to satisfy the constraint condition $u(t,x) \in \mathbb{S}^N$. To see this, suppose that $u : (-T,T) \times X \to \mathbb{S}^N$ is a smooth solution to (1.1)-(1.2). Then, by taking partial derivatives of $|u|^2 = 1$ with respect to t or x, we see $(u, u_t) = 0$, $(u, u_x) = 0$, $(u, u_{xx}) = -|u_x|^2$, $(u, u_{xxx}) = -3(u_{xx}, u_x)$, $(u, u_{xxxx}) = -4(u_{xxx}, u_x) - 3|u_{xx}|^2$, and $(u, u_{xxxxx}) = -5(u_{xxxx}, u_x) - 10(u_{xxx}, u_{xx})$. From them, it follows that

$$
\begin{aligned}
0 = (u, u_t) &= b_1(u, u_{xxxxx}) + 5b_1(u_x, u_{xxxx}) + 10b_1(u_{xxx}, u_{xx}) \\
&\quad + b_2|u_x|^2(u, u_{xxx}) + b_4(u_{xx}, u_x)(u_{xx}, u) + b_6|u_x|^2(u_{xx}, u_x) \\
&= (5b_1 - 5b_1)(u_x, u_{xxxx}) + (10b_1 - 10b_1)(u_{xxx}, u_{xx}) \\
&\quad + (-3b_2 - b_4 + b_6)|u_x|^2(u_{xx}, u_x) \\
&= (-3b_2 - b_4 + b_6)|u_x|^2(u_{xx}, u_x).
\end{aligned}
$$

Therefore, it is reasonable to impose $-3b_2 - b_4 + b_6 = 0$.

The equation (1.1) with $N = 2$ arises in a hierarchy of completely integrable systems containing the following second-order PDE

$$
u_t = u \wedge u_{xx} \qquad (1.3)
$$

where $u = u(t,x) : \mathbb{R} \times X \to \mathbb{S}^2 \subset \mathbb{R}^3$ and $\wedge$ denotes the vector product in $\mathbb{R}^3$. The equation (1.3) arises in several contexts in mathematics. In mathematical physics, (1.3) is known as a one-dimensional classical Heisenberg ferromagnetic spin model ([24]) or Landau-Lifshitz equation with zero dissipation([16]). Also, (1.3) is known to be related to the so-called Da Rios equation which models the motion of a vortex filament in inviscid incompressible fluid in $\mathbb{R}^3$. In geometric analysis, (1.3) is known to be an example of the Schrödinger map (or Schrödinger flow) equation for maps with values into a Riemannian manifold with an almost complex structure. See, e.g.,[3, 4, 14, 20, 27] and references therein. In the context of differentiable dynamical systems, (1.3) is known to be a completely integrable system. See, e.g. [1, 2, 8, 27] and references therein. In particular, Barouch, Fokas and Papageorgiou [2] showed that (1.3) is a bi-Hamiltonian system and constructed the recursion operator generating a hierarchy of integrable evolution equations by the inverse scattering method. Anco and Myrzakulov [1] derived the same bi-Hamiltonian structure, the recursion operator, and conservation laws for each equation in the hierarchy by using a geometric moving frame. The hierarchy actually contains (1.3) as well as the so-called mKdV counterparts. Indeed, following [1], the hierarchy of integrable bi-Hamiltonian flows on $u = u(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{S}^2$ is given explicitly by

$$u_t = (u \wedge D_x - u_x D_x^{-1}(u \wedge u_x, \cdot))^n u_x, \quad n = 0, 1, 2, \dots \tag{1.4}$$

where $\wedge$ is the vector product in $\mathbb{R}^3$, $D_x = \partial/\partial x$, and D_x^{-1} is the inverse. The equation (1.4) corresponding to $n = 1$ is just (1.3), and the one corresponding to $n = 2, 3, 4$ are respectively found to be described as follows:

$$n = 2 : u_t = -u_{xxx} - 3(u_x, u_{xx})u - \frac{3}{2}|u_x|^2 u_x, \tag{1.5}$$

$$n = 3 : u_t = -u \wedge u_{xxxx} - 5(u_x, u_{xx})u \wedge u_x - \frac{5}{2}|u_x|^2 u \wedge u_{xx}, \tag{1.6}$$

$$\begin{aligned} n = 4 : u_t = u_{xxxxx} &+ 5(u_{xxxx}, u_x)u + 10(u_{xxx}, u_{xx})u + \frac{5}{2}|u_x|^2 u_{xxx} \\ &+ 10(u_{xxx}, u_x)u_x + 10(u_{xx}, u_x)u_{xx} + \frac{15}{2}|u_{xx}|^2 u_x \\ &+ \frac{35}{2}|u_x|^2(u_{xx}, u_x)u + \frac{35}{8}|u_x|^4 u_x. \end{aligned} \tag{1.7}$$

The equation (1.7) is actually a particular case of (1.1) where $N = 2$, $b_1 = 1$ and $(b_2, b_4, b_6) = (5/2, 10, 35/2)$ satisfying $3b_2 + b_4 - b_6 = 0$.

1.2. Aim and Known Results

The equation (1.1) is a kind of system of nonlinear dispersive partial differential equations (PDEs) with the geometric constraint condition that the solution takes values in $\mathbb{S}^N$. In general, the solvability of dispersive PDEs is essentially related to the behaviour of lower-order terms in their equations. If there are many types of lower-order derivatives of the solution in the equation, then it may increase the risk of the presence of a bad structure that prevents the classical energy method based on the integration by parts from working. The bad structure is called a loss of derivative, the obstruction of which is to be overcome in some way in order to solve the PDE. See, e.g., [5, 18, 19, 26] and references therein. In particular, to solve our equation (1.1), we need to find a kind of good solvable structure of nonlinear terms including lower-order derivatives. For this purpose, the detailed analysis seems to be required, since (1.1) is a higher-order PDE and the nonlinear structure depends also on the constrained condition $u(t,x) \in \mathbb{S}^N$.

Some dispersive PDE systems with the constraint condition $u(t,x) \in \mathbb{S}^2$ have been investigated from the viewpoint of PDE theory for the last three decades. We recall these previous results briefly. For the second-order $\mathbb{S}^2$-valued model equation (1.3), Sulem, Sulem, and Bardos [24] showed time-local and global existence of a unique smooth solution in a Sobolev space by the classical energy method. One can refer also to [28] for the results. The following third-order dispersive PDE arises in the study of the motion of a vortex filament with the axial flow effect([10]):

$$u_t = u \wedge u_{xx} + a\left(u_{xxx} + 3(u_{xx}, u_x)u + \frac{3}{2}|u_x|^2 u_x\right), \tag{1.8}$$

where $u = u(t,x) : \mathbb{R} \times X \to \mathbb{S}^2$ and $0 \neq a \in \mathbb{R}$ is a constant. For (1.8), Nishiyama and Tani [21, 25] showed time-local and global existence of a unique smooth solution to the initial value problem in a Sobolev space by the classical energy method. The following fourth-order dispersive PDE arises as a higher-order approximation of the Heisenberg ferromagnetic spin model (1.3) ([17]):

$$u_t = u \wedge u_{xx} + a\, u \wedge u_{xxxx} + b\,(u_{xx}, u_x)u \wedge u_x + c\,|u_x|^2 u \wedge u_{xx}, \tag{1.9}$$

where $u = u(t,x) : \mathbb{R} \times X \to \mathbb{S}^2$ and $a, b, c \in \mathbb{R}$ with $a \neq 0$ are constants. It is known that (1.9) arises also in the study of the motion of a vortex filament with the elliptical deformation effect ([9, 10]). For (1.9), the classical energy method does not work by the presence of a loss of derivative, which is an essential difference from the analysis of (1.3) or (1.8). Fortunately however, Guo, Zeng, and Su [12] succeeded to construct a weak time-local solution to the periodic initial value problem in a Sobolev space where $X = \mathbb{T}$ under the additional assumption $b = 2c = 5a$. The proof is based on a modified energy method applying some conservation laws. The conservation laws do not hold in general without the assumption $b = 2c = 5a$. After that, Chihara and Onodera [6] (resp., Onodera [23]) showed time-local existence of a unique smooth solution to the initial value problem in a Sobolev space where $X = \mathbb{R}$ (resp., where $X = \mathbb{T}$) without the assumption on $a \neq 0, b, c$. These proofs were based on the energy method combined with a kind of gauge transformation. These results hold also for (1.5) and (1.6) only by neglecting the term $u \wedge u_{xx}$ in the right hand side of (1.8) and (1.9).

The aim of this paper is to establish the time-local existence result for (1.1)-(1.2) from the viewpoint of PDE theory. The original aim was to find out a solvable structure for (1.4) with arbitrary n. It seemed interesting because the structure of the nonlinear terms becomes complicated as n increases. Particularly, the case of even n seemed easier to be handled than the case of odd n, in that (1.4) for even n is an odd-order dispersive PDE having a constant leading order term. However, it was not easy to write the right hand side of (1.4) for arbitrary n exactly as a polynomial form of the solution and the partial derivatives only from the recursion formula, which was not helpful to study the structure of nonlinear terms. Hence we limit ourselves in the study of the fifth-order equation (1.1) containing (1.4) for $n = 4$ in this paper. Since (1.1) is the first odd-order dispersive PDE with constraint condition $u(t, x) \in \mathbb{S}^N$ which possesses the difficulty of a loss of derivative, we can expect that the study will present an insight on how to solve general odd-order dispersive PDEs in this context containing (1.4) for even n in future.

1.3. Main Results

To state our main results precisely, we introduce some notation used here and hereafter: For integers $m \geq 0$ and $K \geqslant 1$, $H^m(X; \mathbb{R}^K)$ denotes the usual m-th Sobolev space of $\mathbb{R}^K$-valued functions on X equipped with the norm $\|U\|_{H^m} = \left\{\sum_{k=0}^{m} \int_X (\partial_x^k U(x), \partial_x^k U(x)) dx\right\}^{1/2}$ for $U \in H^m(X; \mathbb{R}^K)$, where $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{R}^K$. In particular, $(H^0(X; \mathbb{R}^K), \|\cdot\|_{H^0})$ is denoted by $(L^2(X; \mathbb{R}^K), \|\cdot\|_{L^2})$. For an interval $I \subset \mathbb{R}$ and a Banach space Z, the set of all Z-valued continuous (resp. essentially bounded) functions on I is denoted by $C(I; Z)$ (resp. $L^\infty(I; Z)$). In addition, the set of all continuous maps from a topological space Y into $\mathbb{S}^N$ is denoted by $C(Y, \mathbb{S}^N)$.

We are now in a position to state our main results.

Theorem 1.1. *Let m be an integer satisfying $m \geqslant 8$. For any $u_0 \in C(\mathbb{T}; \mathbb{S}^N)$ satisfying $u_{0x} \in H^m(\mathbb{T}; \mathbb{R}^{N+1})$, there exists a positive constant T depending on $\|u_{0x}\|_{H^8}$ such that* (1.1)-(1.2) *has a unique solution $u \in C([-T, T] \times \mathbb{T}; \mathbb{S}^N)$ satisfying $u_x \in C([-T, T]; H^m(\mathbb{T}; \mathbb{R}^{N+1}))$.*

Theorem 1.1 shows the time-local existence and uniqueness of a solution to the initial value problem (1.1)-(1.2) under the periodic boundary condition where $X = \mathbb{T}$. In fact, Theorem 1.1 is valid also for the case $X = \mathbb{R}$. However, we focus only on the case $X = \mathbb{T}$ in this paper, since the case seems to be more difficult than the case $X = \mathbb{R}$ as is stated in Remark 1 in this section. In addition, Theorem 1.1 is valid for (1.1) which is not necessarily completely integrable, as far as the constraint condition $3a_2 + a_4 - a_6 = 0$ with $b_1 \neq 0$ is satisfied. In other words, we show Theorem 1.1 without full use of the completely integrable structure.

To prove Theorem 1.1, we employ the energy method combined with a kind of gauge transformation following [4, 23]. The key ingredient of the method comes from the following observation: Let u be a smooth solution to (1.1)-(1.2). Then, by the classical energy estimate based on the integration by parts and the Sobolev embedding, we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|_{H^m}^2 \leqslant C_0\|u_x\|_{H^m}^2 &+ \beta_1\int_{\mathbb{T}}(\partial_x u_x, u_x)|\partial_x^{m+1}u_x|^2dx \\ &+ \beta_2\int_{\mathbb{T}}(\partial_x^{m+1}u_x, \partial_x u_x)(\partial_x^{m+1}u_x, u_x)dx \\ &+ \beta_3\int_{\mathbb{T}}(\partial_x^{m+1}u_x, \partial_x^2 u_x)(\partial_x^m u_x, u_x)dx, \qquad (1.10)\end{aligned}$$

where $C_0 > 0, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ are constants depending on $b_1, b_2, \ldots, b_7$ and on m. By the presence of $\partial_x^{m+1}u_x$ in the right hand side of (1.10), we cannot derive the suitable estimate for $\|u_x\|_{H^m}$ directly from (1.10), which means a loss of derivative occurs for (1.1). To overcome the difficulty, we introduce a function V_m by

$$\begin{aligned}&V_m = \partial_x^m u_x - \Phi_1\partial_x^{m-2}u_x - \Phi_2\partial_x^{m-2}u_x - \Phi_3\partial_x^{m-3}u_x, \qquad (1.11)\\ &\Phi_1 = M_1|u_x|^2, \Phi_2 = M_2(\cdot, u_x)u_x, \Phi_3 = M_3(\cdot, \partial_x u_x)u_x,\end{aligned}$$

where $M_1, M_2, M_3 \in \mathbb{R}$ are constants which will be decided later. Then we find that the commutator of the principal part $b_1\partial_x^5$ with the transformation $\partial_x^m u_x \mapsto V_m$ denoted by $[b_1\partial_x^5, I_d - \Phi_1\partial_x^{-2} - \Phi_2\partial_x^{-2} - \Phi_3\partial_x^{-3}]$ generates the term

$$\begin{aligned}&10b_1M_1(\partial_x u_x, u_x)\partial_x^2 + 5b_1M_2\left\{(\partial_x^2\cdot, \partial_x u_x)u_x + (\partial_x^2\cdot, u_x)\partial_x u_x\right\}\\ &\quad + 5b_1M_3(\partial_x\cdot, \partial_x^2 u_x)u_x,\end{aligned}$$

which cancels out the loss of derivative. Indeed, by choosing $M_1 = \beta_1/(10b_1)$, $M_2 = \beta_2/(10b_1)$, and $M_3 = -\beta_3/(5b_1)$, we can obtain

$$\frac{1}{2}\frac{d}{dt}\|V_m\|_{L^2}^2 \leqslant C_0'(\|u_x\|_{H^{m-1}}^2 + \|V_m\|_{L^2}^2) \tag{1.12}$$

for some constant $C_0' > 0$, which guarantees the suitable energy estimate for $\|u_x\|_{H^{m-1}}^2 + \|V_m\|_{L^2}^2$ (instead of $\|u_x\|_{H^m}^2$).

For more details on the proof, some remarks are in order: First, we can make the above argument rigorous to construct a time-local solution by combining a sixth-order parabolic regularization. Second, we apply some geometric properties of $\mathbb{S}^N$-valued functions. We use basic tools of computations in [21, 25] where $\mathbb{S}^2$-valued functions are handled. Since we use properties coming only from $|u| = 1$, any essential difficulties do not occur by replacing $\mathbb{S}^2$ with arbitrary $\mathbb{S}^N$. Third, we show Proposition 2.3 by the estimate for $h = |u|^2 - 1$ in $H^2(\mathbb{T};\mathbb{R})$. Although the argument of the proof basically follows that in [21, 25, 23], the classical energy estimate for $\|h\|_{H^2(\mathbb{T};\mathbb{R})}$ does not work, which is different from that in [21, 25, 23]. We can avoid the difficulty by applying the same type of the transformation as that in (1.11), where $\Phi_2\partial_x^{-2}$ and $\Phi_3\partial_x^{-3}$ are not required. In addition, our argument to show Proposition 2.3 breaks down without the setting for $b_1, b_2, \dots, b_7$ in (1.1). See the computation to obtain (4.9) and Remark 4. Fourth, the uniqueness of the solution can be proved based on the energy estimate for H^3-norm of the difference of the two solutions with the same initial data combined with the same transformation as that in (1.11). To make the argument rigorous, the assumption $m \geqslant 8$ on the smoothness of the solution is applied.

Remark 1. The proof of Theorem1.1 is valid also for the case $X = \mathbb{R}$, in that the transformation $\partial_x^m u_x \mapsto V_m$ in (1.11) can be used without the need for modification. Rather than the obvious fact, restricting ourselves to the case $X = \mathbb{R}$, we have another proof based on the energy method combined with a simpler gauge transformation. More concretely, motivated by [22, 6] and references therein, we introduce a function V_m by

$$\begin{aligned} V_m &:= \partial_x^m u_x - \Phi\partial_x^{m-2}u_x, \\ \Phi(t,x) &:= M\int_0^x (|u_x(t,y)|^2 + |\partial_x u_x(t,y)|^2)\,dy, \end{aligned} \tag{1.13}$$

where $M \in \mathbb{R}$ is a constant which will be decided later. Then we find that the commutator $[b_1\partial_x^5, I_d - \Phi\partial_x^{-2}]$ generates a second order term $-5b_1M(|u_x|^2 +$

$|\partial_x u_x|^2)\partial_x^2$, which is found to dominate the loss of derivatives in (1.10). Indeed, by choosing M so that $b_1 M$ is sufficiently large, we can obtain (1.12). The computaton is simplified compared to that in our proof of Theorem 1.1 which uses (1.11). This is one of the methods to bring out the so-called local smoothing effect inherited to the solution to (1.1) in the setting $x \in \mathbb{R}$. It is to be stressed that the simple method is not valid in the periodic setting $X = \mathbb{T}$. This is essentially because the above local smoothing effect cannot be expected to hold on the compact domain $\mathbb{T}$. Indeed, Φ in (1.13) can not be a map of the space of periodic functions into itself. Therefore, to handle the case $X = \mathbb{T}$, we need to clarify a more essential solvable structure of (1.1), which is successfully achieved by introducing (1.11).

The plan of the paper is as follows: In Section 2, a uniform energy estimate for higher-order derivatives of a sixth-order parabolic regularized approximating solution is derived. In Section 3, the proof of Theorem 1.1 is completed. In Appendix, a detail on how to construct the approximating solution is supplemented.

2. Uniform energy estimate for approximating solutions.

In this section, we first construct a kind of sixth-order parabolic regularized solution which approximates the solution to (1.1)-(1.2), and next derive a uniform energy estimate for higher-order derivatives of the approximating solutions.

For fixed $\varepsilon \in (0,1]$, we consider the following initial value problem:

$$u_t = \varepsilon\, F_6 + F_5, \tag{2.1}$$
$$u(0,x) = u_0(x), \tag{2.2}$$

where $u = u(t,x) : [0,\infty) \times \mathbb{T} \to \mathbb{S}^N$ is the solution, $u_0 = u_0(x) : \mathbb{T} \to \mathbb{S}^N$ is the same initial function as that in (1.1)-(1.2), and F_6 and F_5 are respectively given by

$$F_6 = \partial_x^5 u_x + 6(\partial_x^4 u_x, u_x)u + 15(\partial_x^3 u_x, \partial_x u_x)u + 10|\partial_x^2 u_x|^2 u, \tag{2.3}$$
$$\begin{aligned} F_5 = {} & b_1\partial_x^4 u_x + 5b_1(\partial_x^3 u_x, u_x)u + 10b_1(\partial_x^2 u_x, \partial_x u_x)u + b_2|u_x|^2\partial_x^2 u_x \\ & + b_3(\partial_x^2 u_x, u_x)u_x + b_4(\partial_x u_x, u_x)\partial_x u_x + b_5|\partial_x u_x|^2 u_x \\ & + b_6|u_x|^2(\partial_x u_x, u_x)u + b_7|u_x|^4 u_x. \end{aligned} \tag{2.4}$$

The form of F_5 is just as same as that of the right hand side of (1.1). We can show the following:

Proposition 2.1. *Let $\varepsilon \in (0,1]$ and let m be an integer satisfying $m \geqslant 7$. Then for any $u_0 \in C(\mathbb{T};\mathbb{S}^N)$ satisfying $u_{0x} \in H^m(\mathbb{T};\mathbb{R}^{N+1})$, there exists a positive constant T_ε depending on ε and $\|u_{0x}\|_{H^m}$ such that* (2.1)-(2.2) *has a unique solution $u \in C([0,T_\varepsilon]\times\mathbb{T};\mathbb{S}^N)$ satisfying $u_x \in C([0,T_\varepsilon];H^m(\mathbb{T};\mathbb{R}^{N+1}))$.*

Proposition 2.1 can be proved by combining the following two propositions:

Proposition 2.2. *Under the same assumptions of Proposition* 2.1*, there exists a positive constant T_ε depending on ε and $\|u_{0x}\|_{H^m}$, and exists a unique $u \in C([0,T_\varepsilon]\times\mathbb{T};\mathbb{R}^{N+1})$ satisfying $u_x \in C([0,T_\varepsilon];H^m(\mathbb{T};\mathbb{R}^{N+1}))$ and* (2.1)-(2.2)*.*

Proposition 2.3. *Let u be the $\mathbb{R}^{N+1}$-valued function in Proposition* 2.2*. Then it follows that $|u|=1$ on $[0,T_\varepsilon]\times\mathbb{T}$, that is, $u \in C([0,T_\varepsilon]\times\mathbb{T};\mathbb{S}^N)$.*

By the added term $\varepsilon\, F_6$, (2.1) behaves as a parabolic system with the sixth-order leading term $\varepsilon\partial_x^5 u_x$ and polynomial nonlinearities $F(u)$ of the form

$$u_t = \varepsilon\partial_x^5 u_x + F(u), \tag{2.5}$$

where $F(u)$ consists of nonlinear terms of $u, u_x, \dots, \partial_x^4 u_x$, and has no constant terms and no linear terms except for $b_1\partial_x^4 u_x$. This helps to construct a solution to (2.1)-(2.2) in a relatively simple way. Indeed, by parabolic smoothing properties coming from the leading term $\varepsilon\partial_x^5 u_x$, we can easily show Proposition 2.2 by the contraction mapping argument. Since the argument is standard, we omit the detail.

The role of Proposition 2.3 is to ensure that $u \in C([0,T_\varepsilon]\times\mathbb{T};\mathbb{R}^{N+1})$ constructed in Proposition 2.2 actually takes values in $\mathbb{S}^N$, where the form of F_6 works effectively to obtain a suitable estimate for $h := |u|^2-1$. Although the proof follows the argument in [21, 23, 25], we need a modification of the estimate for $\partial_x^2 h$. The idea of the modification is similar to that used later. The detail on the proof of Proposition 2.3 is presented in Appendix for interested readers.

Next, we derive an energy estimate for the partial derivative of the solution to (2.1)-(2.2) with respect to x uniformly in ε. More concretely, the goal of this section is to show the following.

Proposition 2.4. *Let $m \geqslant 8$ be an integer. For each $\varepsilon \in (0,1]$, let $u^\varepsilon = u^\varepsilon(t,x) : [0,T_\varepsilon]\times\mathbb{T} \to \mathbb{S}^N$ be the solution to* (2.1)-(2.2) *constructed in Proposition* 2.1*. Then, there exists a constant $T = T(\|u_{0x}\|_{H^8}) > 0$ which is independent of $\varepsilon \in (0,1]$ such that $T \leqslant T_\varepsilon$ for all $\varepsilon \in (0,1]$ and $\{u_x^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^{N+1}))$.*

Proof of Proposition 2.4. We introduce a function V_m^ε defined by

$$V_m^\varepsilon = \partial_x^m u_x^\varepsilon - M_1|u_x^\varepsilon|^2\partial_x^{m-2}u_x^\varepsilon - M_2(\partial_x^{m-2}u_x^\varepsilon, u_x^\varepsilon)u_x^\varepsilon - M_3(\partial_x^{m-3}u_x^\varepsilon, \partial_x u_x^\varepsilon)u_x^\varepsilon,$$

where $M_1, M_2, M_3 \in \mathbb{R}$ are constants which will be decided later. For convenience, we set $U_m^\varepsilon = \partial_x^m u_x^\varepsilon$ and $\Lambda(u^\varepsilon) = \Lambda_1(u^\varepsilon) + \Lambda_2(u^\varepsilon) + \Lambda_3(u^\varepsilon)$ where

$$\Lambda_1(u^\varepsilon) = \Phi_1\partial_x^{-2}, \quad \Lambda_2(u^\varepsilon) = \Phi_2\partial_x^{-2}, \quad \Lambda_3(u^\varepsilon) = \Phi_3\partial_x^{-3}, \tag{2.6}$$

$$\Phi_1 = M_1|u_x^\varepsilon|^2, \quad \Phi_2 = M_2(\cdot, u_x^\varepsilon)u_x^\varepsilon, \quad \Phi_3 = M_3(\cdot, \partial_x u_x^\varepsilon)u_x^\varepsilon. \tag{2.7}$$

Then we can write as follows:

$$\begin{aligned} V_m^\varepsilon &= \partial_x^m u_x^\varepsilon - \Phi_1\partial_x^{m-2}u_x^\varepsilon - \Phi_2\partial_x^{m-2}u_x^\varepsilon - \Phi_3\partial_x^{m-3}u_x^\varepsilon \\ &= U_m^\varepsilon - \Lambda(u^\varepsilon)U_m^\varepsilon. \end{aligned} \tag{2.8}$$

Next, we introduce $N_m(u^\varepsilon(t))$, a function of t, defined by

$$N_m(u^\varepsilon(t)) = \left\{\|u_x^\varepsilon(t)\|_{H^{m-1}}^2 + \|V_m^\varepsilon(t)\|_{L^2}^2\right\}^{1/2}. \tag{2.9}$$

Restricting the time interval, $N_m(u^\varepsilon)$ turns out to be equivalent with $\|u_x^\varepsilon\|_{H^m}$. More precisely, we define $T_\varepsilon^\star$ by

$$T_\varepsilon^\star = \sup\left\{T > 0 \mid N_8(u^\varepsilon(t)) \leqslant 2N_8(u_0) \ \text{ for all } \ t \in [0, T]\right\}.$$

Then, by the Sobolev embedding, it turns out that there exists a constant $C = C(\|u_{0x}\|_{H^8}) > 1$ which is independent of ε such that

$$\frac{1}{C}N_m(u^\varepsilon(t)) \leqslant \|u_x^\varepsilon(t)\|_{H^m} \leqslant C\,N_m(u^\varepsilon(t)) \tag{2.10}$$

for all $t \in [0, T_\varepsilon^\star]$. Under the setting, we shall show that there exists a constant $T = T(\|u_{0x}\|_{H^8}) > 0$ which is independent of $\varepsilon \in (0,1]$ and m such that $T_\varepsilon^\star \geqslant T$ uniformly in $\varepsilon \in (0,1]$ and that $\{N_m(u^\varepsilon)\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;\mathbb{R})$. If it is true, then this together with (2.10) implies that $\{u_x^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^{N+1}))$. In what follows, we write $u = u^\varepsilon$, $U_m = U_m^\varepsilon$, $V_m = V_m^\varepsilon$, $\Lambda(u) = \Lambda(u^\varepsilon)$ and $\Lambda_i(u) = \Lambda_i(u^\varepsilon)(i = 1,2,3)$ for simplicity. Any positive constant depending on m, $b_i(i = 1, \dots 7)$, $\|u_{0x}\|_{H^8}$ and not on $\varepsilon \in (0,1]$ will be denoted by the same C. From $m \geqslant 8$ and the Sobolev embedding $H^1(\mathbb{T};\mathbb{R}^{N+1}) \subset C(\mathbb{T};\mathbb{R}^{N+1})$, it follows that

$$\|\partial_x^8 u_x\|_{C([0,T_\varepsilon^\star];L^2)} \leqslant C, \ \|\partial_x^j u_x\|_{C([0,T_\varepsilon^\star]\times\mathbb{T})} \leqslant C \quad (j = 0, 1, \dots, 7). \tag{2.11}$$

In addition, by (2.8), $V_m = U_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$ holds. We will use these properties sometimes without any comments.

To begin with, we compute the PDE satisfied by U_m and next compute the one satisfied by V_m. Suppose that $t \in [0, T_\varepsilon^\star]$. Then, by (2.1), we have

$$\partial_t U_m = \partial_t(\partial_x^m u_x) = \partial_x^{m+1} u_t = \varepsilon \partial_x^{m+1} F_6 + \partial_x^{m+1} F_5. \tag{2.12}$$

After lengthy calculations using the Leibniz rule, (2.3), and (2.4), we obtain

$$\varepsilon \partial_x^{m+1} F_6 = \varepsilon \Big\{ \partial_x^6 U_m + \mathcal{O}\left(\sum_{j=1}^{5} |\partial_x^j U_m| \right) \Big\} + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x| \right), \tag{2.13}$$

$$\begin{aligned}
\partial_x^{m+1} F_5 = b_1 \partial_x^5 U_m &+ d_0(\partial_x^4 U_m, u_x)u \\
&+ d_1(\partial_x^3 U_m, \partial_x u_x)u + d_2(\partial_x^3 U_m, u_x)u_x + d_3 |u_x|^2 \partial_x^3 U_m \\
&+ d_4(\partial_x^2 U_m, \partial_x^2 u_x)u + d_5(\partial_x^2 U_m, \partial_x u_x)u_x + d_6(\partial_x^2 U_m, u_x)\partial_x u_x \\
&+ d_7 |u_x|^2 (\partial_x^2 U_m, u_x)u + d_8(\partial_x u_x, u_x)\partial_x^2 U_m + d_9(\partial_x U_m, \partial_x^2 u_x)u_x \\
&+ d_{10}(\partial_x U_m, \partial_x u_x)\partial_x u_x + d_{11}(\partial_x U_m, u_x)\partial_x^2 u_x \\
&+ d_{12}|u_x|^2(\partial_x U_m, u_x)u_x + d_{13}(\partial_x U_m, \partial_x^3 u_x)u \\
&+ d_{14}(\partial_x U_m, u_x)(\partial_x u_x, u_x)u + d_{15}|u_x|^2(\partial_x U_m, \partial_x u_x)u \\
&+ d_{16}(\partial_x^2 u_x, u_x)\partial_x U_m + d_{17}|\partial_x u_x|^2 \partial_x U_m + d_{18}|u_x|^4 \partial_x U_m + R_m,
\end{aligned} \tag{2.14}$$

where $R_m = \mathcal{O}\left(\sum_{j=0}^m |\partial_x^j u_x| \right)$. Each of $d_0, d_1, \ldots, d_{18}$ is a real constant depending on m and is given as a linear combination of $b_1, b_2, \ldots, b_7$, the exact form of which is not required below. Furthermore, we can rewrite (2.14) as follows:

$$\partial_t U_m = \varepsilon \partial_x^{m+1} F_6 + P(u) U_m + R_m, \tag{2.15}$$

where

$$P(u) = b_1 \partial_x^5 + P_4(u)\partial_x^4 + P_3(u)\partial_x^3 + P_2(u)\partial_x^2 + P_1(u)\partial_x, \tag{2.16}$$

$$P_4(u) = d_0(\cdot, u_x)u, \tag{2.17}$$

$$P_3(u) = d_1(\cdot, \partial_x u_x)u + d_2(\cdot, u_x)u_x + d_3|u_x|^2, \tag{2.18}$$

$$\begin{aligned}
P_2(u) = {} & d_4(\cdot, \partial_x^2 u_x)u + d_5(\cdot, \partial_x u_x)u_x + d_6(\cdot, u_x)\partial_x u_x \\
& + d_7 |u_x|^2 (\cdot, u_x)u + d_8(\partial_x u_x, u_x),
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
P_1(u) = {} & d_9(\cdot, \partial_x^2 u_x)u_x + d_{10}(\cdot, \partial_x u_x)\partial_x u_x + d_{11}(\cdot, u_x)\partial_x^2 u_x + d_{12}|u_x|^2(\cdot, u_x)u_x \\
& + d_{13}(\cdot, \partial_x^3 u_x)u + d_{14}(\cdot, u_x)(\partial_x u_x, u_x)u + d_{15}|u_x|^2(\cdot, \partial_x u_x)u \\
& + d_{16}(\partial_x^2 u_x, u_x) + d_{17}|\partial_x u_x|^2 + d_{18}|u_x|^4.
\end{aligned} \tag{2.20}$$

Then, from (2.8) and (2.15), it follows that

$$
\begin{aligned}
\partial_t V_m &= \partial_t U_m - \partial_t(\Lambda(u)U_m) \\
&= \varepsilon \partial_x^{m+1} F_6 + P(u)U_m + R_m - \partial_t(\Lambda(u)U_m) \\
&= \varepsilon \partial_x^{m+1} F_6 + P(u)V_m + P(u)\Lambda(u)U_m - \partial_t(\Lambda(u)U_m) + R_m. \qquad (2.21)
\end{aligned}
$$

Using the expression (2.21), we have

$$
\frac{1}{2}\frac{d}{dt}\|V_m\|_{L^2}^2 = \int_{\mathbb{T}} (\partial_t V_m, V_m)dx =: \varepsilon I_1 + I_2 + I_3 + I_4, \qquad (2.22)
$$

$$
I_1 = \int_{\mathbb{T}} (\partial_x^{m+1} F_6, V_m)dx, \quad I_2 = \int_{\mathbb{T}} (P(u)V_m, V_m)dx
$$

$$
I_3 = \int_{\mathbb{T}} (P(u)\Lambda(u)U_m - \partial_t(\Lambda(u)U_m), V_m)dx, \quad I_4 = \int_{\mathbb{T}} (R_m, V_m)dx.
$$

We estimate $I_4, \varepsilon I_1, I_2, I_3$ separately. For I_4, $R_m = \mathcal{O}\left(\sum_{j=0}^m |\partial_x^j u_x|\right)$ implies $\|R_m\|_{L^2} \leqslant C\|u_x\|_{H^m}$. Using this and (2.10), we have

$$
\int_{\mathbb{T}} (R_m, V_m)dx \leqslant \|R_m\|_{L^2}\|V_m\|_{L^2} \leqslant CN_m(u)^2. \qquad (2.23)
$$

For εI_1, from (2.13) and $U_m = V_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, it follows that $\widetilde{F_6} = \partial_x^6 V_m + \widetilde{Q}$ where $\widetilde{Q} = \mathcal{O}\left(\sum_{j=1}^5 |\partial_x^j V_m|\right) + \mathcal{O}\left(\sum_{j=0}^m |\partial_x^j u_x|\right)$. Using this, the integration by parts, the Sobolev embedding, and (2.10), we see

$$
\begin{aligned}
\varepsilon I_1 &\leqslant -\varepsilon\|\partial_x^3 V_m\|_{L_2}^2 + \varepsilon \int_{\mathbb{T}} (\widetilde{Q}, V_m)\, dx \\
&\leqslant -\varepsilon\|\partial_x^3 V_m\|_{L_2}^2 + \varepsilon C \sum_{j=1}^5 \|\partial_x^j V_m\|_{L^2}\|V_m\|_{L^2} + \varepsilon C N_m(u)^2.
\end{aligned}
$$

Furthermore, by the Gagliardo-Nirenberg inequality, the Young inequality, and $0 < \varepsilon \leqslant 1$, we obtain

$$
\varepsilon I_1 \leqslant -\frac{\varepsilon}{2}\|\partial_x^3 V_m\|_{L_2}^2 + CN_m(u)^2. \qquad (2.24)
$$

For I_2, by (2.16), we can write

$$
I_2 =: b_1 \int_{\mathbb{T}} (\partial_x^5 V_m, V_m)dx + I_{24} + I_{23} + I_{22} + I_{21}, \qquad (2.25)
$$

$$
I_{2k} = \int_{\mathbb{T}} (P_k(u)\partial_x^4 V_m, V_m)dx \quad (k = 1, 2, 3, 4),
$$

where $P_4(u), \dots, P_1(u)$ are given by (2.17)-(2.20). We compute the right hand side of (2.25). First, by the integration by parts, we see

$$b_1 \int_{\mathbb{T}} (\partial_x^5 V_m, V_m)dx = b_1 \int_{\mathbb{T}} (\partial_x^3 V_m, \partial_x^2 V_m)dx = 0. \tag{2.26}$$

For other terms $I_{24}, \dots, I_{21}$, we estimate by mixing the standard method of the computation as above and some properties coming from the constraint condition $|u| = 1$. We begin with the computation of $I_{24} =: I + II$ where

$$I := \int_{\mathbb{T}} (P_4(u)\partial_x^4 V_m, U_m)dx, \quad II := -\int_{\mathbb{T}} (P_4(u)\partial_x^4 V_m, \Lambda(u)U_m)dx.$$

We first estimate I. By (2.17),

$$I = d_0 \int_{\mathbb{T}} (\partial_x^4 V_m, u_x)(U_m, u)dx.$$

To estimate I, we use the following property

$$\begin{aligned}(U_m, u) = &-(m+1)(\partial_x^{m-1}u_x, u_x) - {}_{m+1}\mathrm{C}_2\,(\partial_x^{m-2}u_x, \partial_x u_x) \\ &- {}_{m+1}\mathrm{C}_3\,(\partial_x^{m-3}u_x, \partial_x^2 u_x) + \mathcal{O}\left(\sum_{j=0}^{m-4} |\partial_x^j u_x|\right).\end{aligned} \tag{2.27}$$

This can be obtained by taking the partial differentiation of $|u|^2 = 1$ with respect to x inductively. (This has been applied also in [21] for a third order nonlinear dispersive PDE modelling the motion of a vortex filament.) By substituting (2.27) and by using the integration by parts, we deduce

$$\begin{aligned}I \leqslant &-d_0(m+1)\int_{\mathbb{T}} (\partial_x^4 V_m, u_x)(\partial_x^{m-1}u_x, u_x)dx \\ &- d_0\,{}_{m+1}\mathrm{C}_2 \int_{\mathbb{T}} (\partial_x^4 V_m, u_x)(\partial_x^{m-2}u_x, \partial_x u_x)dx \\ &- d_0\,{}_{m+1}\mathrm{C}_3 \int_{\mathbb{T}} (\partial_x^4 V_m, u_x)(\partial_x^{m-3}u_x, \partial_x^2 u_x)dx + CN_m(u)^2.\end{aligned}$$

Using the integration by parts again, we deduce

$$
\begin{aligned}
I \leqslant\; & d_0(m+1)\int_{\mathbb{T}}(\partial_x^3 V_m,\partial_x u_x)(\partial_x^{m-1}u_x,u_x)dx \\
& + d_0(m+1)\int_{\mathbb{T}}(\partial_x^3 V_m,u_x)(U_m,u_x)dx \\
& + d_0(m+1)\int_{\mathbb{T}}(\partial_x^3 V_m,u_x)(\partial_x^{m-1}u_x,\partial_x u_x)dx \\
& + d_0\,{}_{m+1}\mathrm{C}_2\int_{\mathbb{T}}(\partial_x^3 V_m,\partial_x u_x)(\partial_x^{m-2}u_x,\partial_x u_x)dx \\
& + d_0\,{}_{m+1}\mathrm{C}_2\int_{\mathbb{T}}(\partial_x^3 V_m,u_x)(\partial_x^{m-1}u_x,\partial_x u_x)dx \\
& + d_0\,{}_{m+1}\mathrm{C}_2\int_{\mathbb{T}}(\partial_x^3 V_m,u_x)(\partial_x^{m-2}u_x,\partial_x^2 u_x)dx \\
& + d_0\,{}_{m+1}\mathrm{C}_3\int_{\mathbb{T}}(\partial_x^3 V_m,u_x)(\partial_x^{m-2}u_x,\partial_x^2 u_x)dx + CN_m(u)^2.
\end{aligned}
\tag{2.28}
$$

By the repeated use of the integration by parts as above, we arrive at

$$
\begin{aligned}
I \leqslant\; & \alpha_{11}\int_{\mathbb{T}}(\partial_x V_m,\partial_x^2 u_x)(U_m,u_x)dx + \alpha_{12}\int_{\mathbb{T}}(\partial_x V_m,\partial_x u_x)(\partial_x U_m,u_x)dx \\
& + \alpha_{13}\int_{\mathbb{T}}(\partial_x V_m,\partial_x u_x)(U_m,\partial_x u_x)dx + \alpha_{14}\int_{\mathbb{T}}(\partial_x^2 V_m,u_x)(\partial_x U_m,u_x)dx \\
& + \alpha_{15}\int_{\mathbb{T}}(\partial_x V_m,u_x)(\partial_x U_m,\partial_x u_x)dx + \alpha_{16}\int_{\mathbb{T}}(\partial_x V_m,u_x)(U_m,\partial_x^2 u_x)dx \\
& + CN_m(u)^2,
\end{aligned}
\tag{2.29}
$$

where $\alpha_{11},\dots,\alpha_{16}$ are real constants depending on d_0 and m but are independent of (M_1,M_2,M_3). Although the fourth term of the right hand side of (2.29) still contains $\partial_x^2 V_m$, by the integration by parts and $U_m = V_m + \mathcal{O}(|\partial_x^{m-2}u_x| +$

$|\partial_x^{m-3}u_x|)$, we deduce

$$
\begin{aligned}
&\alpha_{14}\int_{\mathbb{T}}(\partial_x^2 V_m, u_x)(\partial_x U_m, u_x)dx \\
&\leqslant \alpha_{14}\int_{\mathbb{T}}(\partial_x^2 V_m, u_x)(\partial_x V_m, u_x)dx + CN_m(u)^2 \\
&= \frac{\alpha_{14}}{2}\int_{\mathbb{T}}(\partial_x^2 V_m, u_x)(\partial_x V_m, u_x)dx - \frac{\alpha_{14}}{2}\int_{\mathbb{T}}(\partial_x V_m, u_x)(\partial_x^2 V_m, u_x)dx \\
&\quad - \frac{\alpha_{14}}{2}\int_{\mathbb{T}}(\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx - \frac{\alpha_{14}}{2}\int_{\mathbb{T}}(\partial_x V_m, u_x)(\partial_x V_m, \partial_x u_x)dx \\
&\quad + CN_m(u)^2 \\
&= -\alpha_{14}\int_{\mathbb{T}}(\partial_x V_m, u_x)(\partial_x V_m, \partial_x u_x)dx + CN_m(u)^2. \qquad (2.30)
\end{aligned}
$$

Furthermore, substituting (2.30) into (2.29) and using $U_m = V_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we deduce

$$
\begin{aligned}
I \leqslant\ &\alpha_{11}\int_{\mathbb{T}}(\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx \\
&+ (\alpha_{12} - \alpha_{14} + \alpha_{15})\int_{\mathbb{T}}(\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx \\
&+ \alpha_{13}\int_{\mathbb{T}}(\partial_x V_m, \partial_x u_x)(V_m, \partial_x u_x)dx + \alpha_{16}\int_{\mathbb{T}}(\partial_x V_m, u_x)(V_m, \partial_x^2 u_x)dx \\
&+ CN_m(u)^2. \qquad (2.31)
\end{aligned}
$$

Here, the third term of the right hand side of (2.31) is bounded by $CN_m(u)^2$ by the integration by parts as we show (2.30). The fourth term of the right hand side of (2.31) is estimated by the integration by parts as follows:

$$
\alpha_{16}\int_{\mathbb{T}}(\partial_x V_m, u_x)(V_m, \partial_x^2 u_x)dx \leqslant -\alpha_{16}\int_{\mathbb{T}}(\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx + CN_m(u)^2.
$$

Using them, we have

$$
\begin{aligned}
I \leqslant CN_m(u)^2 &+ (\alpha_{11} - \alpha_{16})\int_{\mathbb{T}}(\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx \\
&+ (\alpha_{12} - \alpha_{14} + \alpha_{15})\int_{\mathbb{T}}(\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx.
\end{aligned}
$$

Note that the second and the third terms of the right hand side of above cannot be handled as above. This is a reason we introduce (2.8).
Next we estimate II. Recalling (2.17) and using $(u_x, u) = 0$ and $\Lambda U_m = M_1|u_x|^2\partial_x^{m-2}u_x + M_2(\partial_x^{m-2}u_x, u_x)u_x + M_3(\partial_x^{m-3}u_x, \partial_x u_x)u_x$, we see

$$II = -d_0 M_1 \int_{\mathbb{T}} |u_x|^2(\partial_x^4 V_m, u_x)(\partial_x^{m-2}u_x, u)\, dx.$$

We use (2.27) replacing m with $m-2$ to see

$$(\partial_x^{m-2}u_x, u) = -(m-1)(\partial_x^{m-3}u_x, u_x) + \mathcal{O}\left(\sum_{j=0}^{m-4} |\partial_x^j u_x|\right).$$

Substituting this relation and using the integration by parts and $U_m = V_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we deduce

$$\begin{aligned}
II &\leqslant d_0(m-1)M_1 \int_{\mathbb{T}} |u_x|^2(\partial_x^4 V_m, u_x)(\partial_x^{m-3}u_x, u_x)\, dx + CN_m(u)^2 \\
&\leqslant -d_0(m-1)M_1 \int_{\mathbb{T}} |u_x|^2(\partial_x V_m, u_x)(U_m, u_x)\, dx + CN_m(u)^2 \\
&\leqslant -d_0(m-1)M_1 \int_{\mathbb{T}} |u_x|^2(\partial_x V_m, u_x)(V_m, u_x)\, dx + CN_m(u)^2.
\end{aligned}$$

Furthermore, it follows from integration by parts

$$\begin{aligned}
&\int_{\mathbb{T}} |u_x|^2(\partial_x V_m, u_x)(V_m, u_x)\, dx \\
&= -\int_{\mathbb{T}} (\partial_x u_x, u_x)(V_m, u_x)(V_m, u_x)\, dx - \int_{\mathbb{T}} |u_x|^2(V_m, \partial_x u_x)(V_m, u_x)\, dx \\
&\leqslant CN_m(u)^2. \qquad (2.32)
\end{aligned}$$

Therefore, we obtain $II \leqslant CN_m(u)^2$. Combining the estimate for I and that for II, we obtain

$$\begin{aligned}
I_{24} \leqslant CN_m(u)^2 &+ (\alpha_{11} - \alpha_{16}) \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x) dx \\
&+ (\alpha_{12} - \alpha_{14} + \alpha_{15}) \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x) dx. \qquad (2.33)
\end{aligned}$$

Next we estimate I_{23}. By (2.18) and $V_m = U_m - \Lambda(u)U_m$, we have

$$
\begin{aligned}
I_{23} &= d_1 \int_{\mathbb{T}} (\partial_x^3 V_m, \partial_x u_x)(V_m, u)dx + d_2 \int_{\mathbb{T}} (\partial_x^3 V_m, u_x)(V_m, u_x)dx \\
&\quad + d_3 \int_{\mathbb{T}} |u_x|^2 (\partial_x^3 V_m, V_m)dx \\
&= d_1 \int_{\mathbb{T}} (\partial_x^3 V_m, \partial_x u_x)(U_m, u)dx - d_1 \int_{\mathbb{T}} (\partial_x^3 V_m, \partial_x u_x)(\Lambda(u)U_m, u)dx \\
&\quad + d_2 \int_{\mathbb{T}} (\partial_x^3 V_m, u_x)(V_m, u_x)dx + d_3 \int_{\mathbb{T}} |u_x|^2 (\partial_x^3 V_m, V_m)dx. \qquad (2.34)
\end{aligned}
$$

By lengthy calculations using the same argument as we obtain (2.33) from (2.28), we can obtain

$$
\begin{aligned}
I_{23} &\leqslant C N_m(u)^2 + \alpha_{17} \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x V_m|^2 dx \\
&\quad + \alpha_{18} \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx + \alpha_{19} \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx,
\end{aligned}
\tag{2.35}
$$

where $\alpha_{17}, \alpha_{18}, \alpha_{19}$ are real constants depending on d_1, d_2, d_3, m, but are independent of (M_1, M_2, M_3). Note that the second term of the right hand side of (2.35) comes from the fourth term of the right hand side of (2.34). In the same way, we can obtain

$$
\begin{aligned}
& I_{22} + I_{21} \\
& \leqslant C N_m(u)^2 + \alpha_{20} \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x V_m|^2 dx \\
& \quad + \alpha_{21} \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx + \alpha_{22} \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx,
\end{aligned}
\tag{2.36}
$$

where $\alpha_{20}, \alpha_{21}, \alpha_{22}$ are real constants depending on m but not on (M_1, M_2, M_3), and each of the constants is a linear combination of $d_4, \ldots, d_{18}$. Substituting (2.33), (2.35), and (2.36) into (2.25), we obtain

$$
\begin{aligned}
I_2 &\leqslant C N_m(u)^2 + \beta_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x V_m|^2 dx \\
&\quad + \beta_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx + \beta_3 \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx,
\end{aligned}
\tag{2.37}
$$

where $\beta_1=\alpha_{17}+\alpha_{20}$, $\beta_2=\alpha_{12}-\alpha_{14}+\alpha_{15}+\alpha_{18}+\alpha_{21}$, $\beta_3=\alpha_{11}-\alpha_{16}+\alpha_{19}+\alpha_{22}$.

For I_3, we begin with the computation of

$$P(u)\Lambda(u)U_m-\partial_t(\Lambda(u)U_m)=\sum_{i=1}^{3}\{P(u)\Lambda_i(u)U_m-\partial_t(\Lambda_i(u)U_m)\}, \quad (2.38)$$

which plays a crucial part in the proof. First, we consider $P(u)\Lambda_1(u)U_m-\partial_t(\Lambda_1(u)U_m)$. Noting $\partial_x^m u_x$ and $\partial_x^{m-1}u_x$ are not included in $\Lambda_1(u)U_m$, we compute using (2.16)-(2.20) to deduce

$$\begin{aligned}
P(u)\Lambda_1(u)U_m &= (b_1\partial_x^5+P_4(u)\partial_x^4+P_3(u)\partial_x^3)\Lambda_1(u)U_m+\mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right)\\
&= b_1\partial_x^5(\Lambda_1(u)U_m)+d_0(\partial_x^4(\Lambda_1(u)U_m),u_x)u\\
&\quad +d_1(\partial_x^3(\Lambda_1(u)U_m),\partial_x u_x)u+d_2(\partial_x^3(\Lambda_1(u)U_m),u_x)u_x\\
&\quad +d_3|u_x|^2\partial_x^3(\Lambda_1(u)U_m)+\mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \quad (2.39)
\end{aligned}$$

By the definition of $\Lambda_1(u)$ and $V_m=U_m+\mathcal{O}(|\partial_x^{m-2}u_x|+|\partial_x^{m-3}u_x|)$, we see

$$\begin{aligned}
\partial_x^3(\Lambda_1(u)U_m) &= M_1|u_x|^2\partial_x^{m+1}u_x+\mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right)\\
&= M_1|u_x|^2\partial_x V_m+\mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right), \quad (2.40)\\
\partial_x^4(\Lambda_1(u)U_m) &= M_1|u_x|^2\partial_x^{m+2}u_x+8M_1(\partial_x u_x,u_x)\partial_x^{m+1}u_x+\mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right)\\
&= M_1|u_x|^2\partial_x^2 V_m+8M_1(\partial_x u_x,u_x)\partial_x V_m+\mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \quad (2.41)
\end{aligned}$$

Substituting (2.40) and (2.41) into (2.39), we have

$$
\begin{aligned}
P(u)\Lambda_1(u)U_m = {} & b_1\partial_x^5(\Lambda_1(u)U_m) + d_0M_1|u_x|^2(\partial_x^2V_m, u_x)u \\
& + 40b_1M_1(\partial_xu_x, u_x)(\partial_xV_m, u_x)u + d_1M_1|u_x|^2(\partial_xV_m, \partial_xu_x)u \\
& + d_2M_1|u_x|^2(\partial_xV_m, u_x)u_x + d_3M_1|u_x|^4\partial_xV_m \\
& + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
\end{aligned}
\tag{2.42}
$$

Furthermore, we compute

$$
\partial_t(\Lambda_1(u)U_m) = 2M_1(\partial_xu_t, u_x)\partial_x^{m-2}u_x + M_1|u_x|^2\partial_t\partial_x^{m-2}u_x.
$$

Since $u_t = \varepsilon\,\partial_x^5u_x + \cdots \in C([0, T_\varepsilon^\star]; H^3(\mathbb{T}; \mathbb{R}^{N+1}))$ follows from $m \geqslant 8$,

$$
|(\partial_xu_t, u_x)| \leqslant \|\partial_xu_t\|_{L^\infty}\|u_x\|_{L^\infty} \leqslant C(\|u_x\|_{C([0,T_\varepsilon^\star];H^7)}) < +\infty,
$$

which yields

$$
\partial_t(\Lambda_1(u)U_m) = M_1|u_x|^2\partial_t\partial_x^{m-2}u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
\tag{2.43}
$$

In addition, in the same way as we obtain (2.14) and (2.13), we have

$$
\begin{aligned}
\partial_t\partial_x^{m-2}u_x = {} & \varepsilon\,F' + b_1\partial_x^3U_m + d_0'(\partial_x^2U_m, u_x)u + d_1'(\partial_xU_m, \partial_xu_x)u \\
& + d_2'(\partial_xU_m, u_x)u_x + d_3'|u_x|^2\partial_xU_m + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right) \\
= {} & \varepsilon\,F' + b_1\partial_x^3U_m + d_0'(\partial_x^2V_m, u_x)u + d_1'(\partial_xV_m, \partial_xu_x)u \\
& + d_2'(\partial_xV_m, u_x)u_x + d_3'|u_x|^2\partial_xV_m + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
\end{aligned}
\tag{2.44}
$$

Here $d_0', \ldots, d_3'$ are real constants depending on d_0, d_1, d_2, d_3. And

$$
F' = \partial_x^4U_m + \mathcal{O}\left(\sum_{j=1}^{3}|\partial_x^jU_m|\right) + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
\tag{2.45}
$$

Substituting (2.44) into (2.43), and noting $\Lambda_1(u)\partial_x^5 U_m = M_1|u_x|^2\partial_x^3 U_m$, we obtain

$$\begin{aligned}\partial_t(\Lambda_1(u)U_m) &= \varepsilon\, M_1|u_x|^2F' + b_1\Lambda_1(u)(\partial_x^5 U_m) + d_0' M_1|u_x|^2(\partial_x^2 V_m, u_x)u \\ &\quad + d_1' M_1|u_x|^2(\partial_x V_m, \partial_x u_x)u + d_2' M_1|u_x|^2(\partial_x V_m, u_x)u_x \\ &\quad + d_3' M_1|u_x|^4\partial_x V_m + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \end{aligned} \tag{2.46}$$

Combining (2.42) and (2.46), we obtain

$$\begin{aligned} &P(u)\Lambda_1(u)U_m - \partial_t(\Lambda_1(u)U_m) \\ &= -\varepsilon\, M_1|u_x|^2F' + b_1\left[\partial_x^5, \Lambda_1(u)\right]U_m + (d_0 - d_0')M_1|u_x|^2(\partial_x^2 V_m, u_x)u \\ &\quad + 40b_1M_1(\partial_x u_x, u_x)(\partial_x V_m, u_x)u + (d_1 - d_1')M_1|u_x|^2(\partial_x V_m, \partial_x u_x)u \\ &\quad + (d_2 - d_2')M_1|u_x|^2(\partial_x V_m, u_x)u_x + (d_3 - d_3')M_1|u_x|^4\partial_x V_m \\ &\quad + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \end{aligned} \tag{2.47}$$

Here, by the Leibniz rule and $U_m = V_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we see

$$\begin{aligned} &\left[\partial_x^5, \Lambda_1(u)\right]U_m = M_1\partial_x^5\left\{|u_x|^2\partial_x^{-2}U_m\right\} - M_1|u_x|^2\partial_x^{-2}\partial_x^5 U_m \\ &= M_1|u_x|^2\partial_x^3 U_m + 5M_1\partial_x\left\{|u_x|^2\right\}\partial_x^2 U_m + 10M_1\partial_x^2\left\{|u_x|^2\right\}\partial_x U_m \\ &\quad + M_1\sum_{j=3}^{5}{}_5\mathrm{C}_j\,\partial_x^j\left\{|u_x|^2\right\}\partial_x^{5-j}\partial_x^{-2}U_m - M_1|u_x|^2\partial_x^3 U_m \\ &= 10M_1(\partial_x u_x, u_x)\partial_x^2 U_m + 10M_1\partial_x^2\left\{|u_x|^2\right\}\partial_x U_m + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right) \\ &= 10M_1(\partial_x u_x, u_x)\partial_x^2 V_m + 10M_1\partial_x^2\left\{|u_x|^2\right\}\partial_x V_m + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right), \end{aligned} \tag{2.48}$$

Substituting (2.48) into (2.47), we obtain

$$
\begin{aligned}
&P(u)\Lambda_1(u)U_m - \partial_t(\Lambda_1(u)U_m) \\
&= -\varepsilon\, M_1|u_x|^2 F' + 10b_1 M_1(\partial_x u_x, u_x)\partial_x^2 V_m + 10 M_1 \partial_x^2 \left\{|u_x|^2\right\} \partial_x V_m \\
&\quad + (d_0 - d_0')M_1|u_x|^2(\partial_x^2 V_m, u_x)u + 40b_1 M_1(\partial_x u_x, u_x)(\partial_x V_m, u_x)u \\
&\quad + (d_1 - d_1')M_1|u_x|^2(\partial_x V_m, \partial_x u_x)u + (d_2 - d_2')M_1|u_x|^2(\partial_x V_m, u_x)u_x \\
&\quad + (d_3 - d_3')M_1|u_x|^4 \partial_x V_m + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right). \qquad (2.49)
\end{aligned}
$$

Second, we observe that $P(u)\Lambda_2(u)U_m - \partial_t(\Lambda_2(u)U_m)$. Noting $\partial_x^m u_x$ and $\partial_x^{m-1}u_x$ are not included in $\Lambda_2(u)U_m$, we compute using (2.16)-(2.20) to deduce

$$
\begin{aligned}
P(u)\Lambda_2(u)U_m &= (b_1\partial_x^5 + P_4(u)\partial_x^4 + P_3(u)\partial_x^3)\Lambda_2(u)U_m + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right) \\
&= b_1\partial_x^5(\Lambda_2(u)U_m) + d_0(\partial_x^4(\Lambda_2(u)U_m), u_x)u \\
&\quad + d_1(\partial_x^3(\Lambda_2(u)U_m), \partial_x u_x)u + d_2(\partial_x^3(\Lambda_2(u)U_m), u_x)u_x \\
&\quad + d_3|u_x|^2\partial_x^3(\Lambda_2(u)U_m) + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right). \qquad (2.50)
\end{aligned}
$$

By the definition of $\Lambda_2(u)$ and $V_m = U_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we have

$$
\begin{aligned}
\partial_x^3(\Lambda_2(u)U_m) &= M_2(\partial_x^{m+1}u_x, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right) \\
&= M_2(\partial_x V_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right), \qquad (2.51)
\end{aligned}
$$

$$
\begin{aligned}
\partial_x^4(\Lambda_2(u)U_m) &= M_2(\partial_x^{m+2}u_x, u_x)u_x + 4M_2(\partial_x^{m+1}u_x, \partial_x u_x)u_x \\
&\quad + 4M_2(\partial_x^{m+1}u_x, u_x)\partial_x u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right) \\
&= M_2(\partial_x^2 V_m, u_x)u_x + 4M_2(\partial_x V_m, \partial_x u_x)u_x \\
&\quad + 4M_2(\partial_x V_m, u_x)\partial_x u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right). \qquad (2.52)
\end{aligned}
$$

Substituting (2.51) and (2.52) into (2.50), we have

$$
\begin{aligned}
&P(u)\Lambda_2(u)U_m \\
&= b_1\partial_x^5(\Lambda_2(u)U_m) + d_0M_2|u_x|^2(\partial_x^2V_m, u_x)u \\
&\quad + 4d_0M_2|u_x|^2(\partial_xV_m, \partial_xu_x)u + (4d_0+d_1)M_2(\partial_xV_m, u_x)(\partial_xu_x, u_x)u \\
&\quad + (d_2+d_3)M_2|u_x|^2(\partial_xV_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right)
\end{aligned}
\tag{2.53}
$$

Furthermore, we compute

$$
\begin{aligned}
\partial_t(\Lambda_2(u)U_m) &= M_2(\partial_t\partial_x^{m-2}u_x, u_x)u_x + M_2(\partial_x^{m-2}u_x, \partial_xu_t)u_x \\
&\quad + M_2(\partial_x^{m-2}u_x, u_x)\partial_xu_t.
\end{aligned}
$$

Since $u_t = \varepsilon\,\partial_x^5u_x + \cdots \in C([0, T_\varepsilon^\star]; H^3(\mathbb{T}; \mathbb{R}^{N+1}))$ follows from $m \geqslant 8$, we see

$$
\begin{aligned}
|\partial_x^{m-2}u_x||u_x||\partial_xu_t| &\leqslant \|\partial_xu_t\|_{L^\infty}\|u_x\|_{L^\infty}|\partial_x^{m-2}u_x| \\
&\leqslant C(\|u_x\|_{C([0,T_\varepsilon^\star];H^7)})|\partial_x^{m-2}u_x|.
\end{aligned}
$$

Therefore, we see

$$
\partial_t(\Lambda_2(u)U_m) = M_2(\partial_t\partial_x^{m-2}u_x, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
$$

Furthermore, by (2.44), (2.45) and $\Lambda_2(u)\partial_x^5U_m = M_2(\partial_x^3U_m, u_x)u_x$, we obtain

$$
\begin{aligned}
&\partial_t(\Lambda_2(u)U_m) \\
&= \varepsilon\,M_2(F', u_x)u_x + b_1\Lambda_2(u)(\partial_x^5U_m) + d_0'M_2(\partial_x^2V_m, u_x)(u_x, u)u_x \\
&\quad + d_1'M_2(\partial_xV_m, \partial_xu_x)(u_x, u)u_x + d_2'M_2|u_x|^2(\partial_xV_m, u_x)u_x \\
&\quad + d_3'M_2|u_x|^2(\partial_xV_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
\end{aligned}
\tag{2.54}
$$

Since $(u_x, u) = 0$ follows from $|u|^2 = 1$, we have

$$
\begin{aligned}
\partial_t(\Lambda_2(u)U_m) &= \varepsilon\,M_2(F', u_x)u_x + b_1\Lambda_2(u)(\partial_x^5U_m) \\
&\quad + (d_2'+d_3')M_2|u_x|^2(\partial_xV_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^ju_x|\right).
\end{aligned}
\tag{2.55}
$$

Combining (2.53) and (2.55), we obtain

$$
\begin{aligned}
& P(u)\Lambda_2(u)U_m - \partial_t(\Lambda_2(u)U_m) \\
&= -\varepsilon\, M_2(F', u_x)u_x + b_1 \left[\partial_x^5, \Lambda_2(u)\right] U_m + d_0 M_2 |u_x|^2 (\partial_x^2 V_m, u_x)u \\
&\quad + 4d_0 M_2 |u_x|^2 (\partial_x V_m, \partial_x u_x)u + (4d_0 + d_1) M_2 (\partial_x V_m, u_x)(\partial_x u_x, u_x)u \\
&\quad + (d_2 + d_3 - d_2' - d_3') M_2 |u_x|^2 (\partial_x V_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right). \qquad (2.56)
\end{aligned}
$$

Here, by the Leibniz rule and $U_m = V_m + \mathcal{O}(\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we see

$$
\begin{aligned}
& \left[\partial_x^5, \Lambda_2(u)\right] U_m = M_2 \partial_x^5 \left\{(\partial_x^{-2} U_m, u_x)u_x\right\} - M_2(\partial_x^{-2}\partial_x^5 U_m, u_x)u_x \\
&= 5M_2(\partial_x^2 U_m, \partial_x u_x)u_x + 5M_2(\partial_x^2 U_m, u_x)\partial_x u_x + 10M_2(\partial_x U_m, \partial_x^2 u_x)u_x \\
&\quad + 20M_2(\partial_x U_m, \partial_x u_x)\partial_x u_x + 10M_2(\partial_x U_m, u_x)\partial_x^2 u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right) \\
&= 5M_2(\partial_x^2 V_m, \partial_x u_x)u_x + 5M_2(\partial_x^2 V_m, u_x)\partial_x u_x + 10M_2(\partial_x V_m, \partial_x^2 u_x)u_x \\
&\quad + 20M_2(\partial_x V_m, \partial_x u_x)\partial_x u_x + 10M_2(\partial_x V_m, u_x)\partial_x^2 u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right).
\end{aligned}
\qquad (2.57)
$$

Substituting (2.57) into (2.56), we obtain

$$
\begin{aligned}
& P(u)\Lambda_2(u)U_m - \partial_t(\Lambda_2(u)U_m) \\
&= -\varepsilon\, M_2(F', u_x)u_x + 5b_1 M_2(\partial_x^2 V_m, \partial_x u_x)u_x + 5b_1 M_2(\partial_x^2 V_m, u_x)\partial_x u_x \\
&\quad + 10b_1 M_2(\partial_x V_m, \partial_x^2 u_x)u_x + 20b_1 M_2(\partial_x V_m, \partial_x u_x)\partial_x u_x \\
&\quad + 10b_1 M_2(\partial_x V_m, u_x)\partial_x^2 u_x + d_0 M_2 |u_x|^2 (\partial_x^2 V_m, u_x)u \\
&\quad + 4d_0 M_2 |u_x|^2 (\partial_x V_m, \partial_x u_x)u + (4d_0 + d_1) M_2 (\partial_x V_m, u_x)(\partial_x u_x, u_x)u \\
&\quad + (d_2 + d_3 - d_2' - d_3') M_2 |u_x|^2 (\partial_x V_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right). \qquad (2.58)
\end{aligned}
$$

Third, we compute $P(u)\Lambda_3(u)U_m - \partial_t(\Lambda_3(u)U_m)$. Noting $\partial_x^m u_x$, $\partial_x^{m-1}u_x$ and

$\partial_x^{m-2}u_x$ are not included in $\Lambda_3(u)U_m$, we use (2.16)-(2.20) to deduce

$$\begin{aligned}P(u)\Lambda_3(u)U_m &= (b_1\partial_x^5 + P_4(u)\partial_x^4)\Lambda_3(u)U_m + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right)\\ &= b_1\partial_x^5(\Lambda_3(u)U_m) + d_0(\partial_x^4(\Lambda_3(u)U_m), u_x)u + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right).\end{aligned} \tag{2.59}$$

By the definition of $\Lambda_3(u)$ and $V_m = U_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we see

$$\begin{aligned}\partial_x^4(\Lambda_3(u)U_m) &= M_3(\partial_x^{m+1}u_x, \partial_x u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right)\\ &= M_3(\partial_x V_m, \partial_x u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right).\end{aligned} \tag{2.60}$$

Substituting (2.60) into (2.59), we have

$$\begin{aligned}&P(u)\Lambda_3(u)U_m\\ &= b_1\partial_x^5(\Lambda_3(u)U_m) + d_0 M_3|u_x|^2(\partial_x V_m, \partial_x u_x)u + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right)\end{aligned} \tag{2.61}$$

Furthermore, we compute

$$\begin{aligned}\partial_t(\Lambda_3(u)U_m) = M_3(\partial_t\partial_x^{m-3}u_x, \partial_x u_x)u_x + M_3(\partial_x^{m-3}u_x, \partial_x^2 u_t)u_x\\ + M_3(\partial_x^{m-3}u_x, \partial_x u_x)\partial_x u_t.\end{aligned}$$

Since $u_t = \varepsilon\,\partial_x^5 u_x + \cdots \in C([0,T_\varepsilon^\star]; H^3(\mathbb{T};\mathbb{R}^{N+1}))$ follows from $m \geqslant 8$, we see

$$\begin{aligned}|(\partial_x^{m-3}u_x, \partial_x^2 u_t)u_x| &\leqslant \|\partial_x^2 u_t\|_{L^\infty}\|u_x\|_{L^\infty}|\partial_x^{m-3}u_x|\\ &\leqslant C(\|u_x\|_{C([0,T_\varepsilon^\star];H^8)})|\partial_x^{m-3}u_x|,\\ |(\partial_x^{m-3}u_x, \partial_x u_x)\partial_x u_t| &\leqslant \|\partial_x u_t\|_{L^\infty}\|u_x\|_{L^\infty}|\partial_x^{m-3}u_x|\\ &\leqslant C(\|u_x\|_{C([0,T_\varepsilon^\star];H^7)})|\partial_x^{m-3}u_x|.\end{aligned}$$

Therefore, we see

$$\partial_t(\Lambda_3(u)U_m) = M_3(\partial_t\partial_x^{m-3}u_x, \partial_x u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \tag{2.62}$$

In addition, in the same way as we obtain (2.14) and (2.13), we have

$$\begin{aligned}\partial_t\partial_x^{m-3}u_x &= \varepsilon\, F'' + b_1\partial_x^2 U_m + d_0''(\partial_x U_m, u_x)u + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right) \\ &= \varepsilon\, F'' + b_1\partial_x^2 U_m + d_0''(\partial_x V_m, u_x)u + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right), \qquad (2.63)\end{aligned}$$

$$F'' = \partial_x^3 U_m + \mathcal{O}\left(\sum_{j=1}^{2}|\partial_x^j U_m|\right) + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right), \qquad (2.64)$$

where d_0'' is a real constant depending on d_0. Substituting (2.63) into (2.62), and noting $\Lambda_3(u)\partial_x^5 U_m = M_3(\partial_x^2 U_m, \partial_x u_x)u_x$, we obtain

$$\begin{aligned}\partial_t(\Lambda_3(u)U_m) &= \varepsilon\, M_3(F'', \partial_x u_x)u_x + b_1\Lambda_3(u)(\partial_x^5 U_m) \\ &\quad + d_0'' M_3(\partial_x V_m, u_x)(\partial_x u_x, u)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right).\end{aligned}$$

Since $(\partial_x u_x, u) = -|u_x|^2$ follows from $|u|^2 = 1$, we have

$$\begin{aligned}\partial_t(\Lambda_3(u)U_m) &= \varepsilon\, M_3(F'', \partial_x u_x)u_x + b_1\Lambda_3(u)(\partial_x^5 U_m) \\ &\quad - d_0'' M_3|u_x|^2(\partial_x V_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \qquad (2.65)\end{aligned}$$

Combining (2.61) and (2.65), we obtain

$$\begin{aligned}&P(u)\Lambda_3(u)U_m - \partial_t(\Lambda_3(u)U_m) \\ &= -\varepsilon\, M_3(F'', \partial_x u_x)u_x + b_1\left[\partial_x^5, \Lambda_3(u)\right]U_m + d_0 M_3|u_x|^2(\partial_x V_m, \partial_x u_x)u \\ &\quad + d_0'' M_3|u_x|^2(\partial_x V_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \qquad (2.66)\end{aligned}$$

Here, by the Leibniz rule and $U_m = V_m + \mathcal{O}(|\partial_x^{m-2}u_x| + |\partial_x^{m-3}u_x|)$, we have

$$\begin{aligned}&\left[\partial_x^5, \Lambda_3(u)\right]U_m = M_3\partial_x^5\left\{(\partial_x^{-3}U_m, \partial_x u_x)u_x\right\} - M_3(\partial_x^{-3}\partial_x^5 U_m, \partial_x u_x)u_x \\ &= 5M_3(\partial_x U_m, \partial_x^2 u_x)u_x + 5M_3(\partial_x U_m, \partial_x u_x)\partial_x u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right) \\ &= 5M_3(\partial_x V_m, \partial_x^2 u_x)u_x + 5M_3(\partial_x V_m, \partial_x u_x)\partial_x u_x + \mathcal{O}\left(\sum_{j=0}^{m}|\partial_x^j u_x|\right). \qquad (2.67)\end{aligned}$$

Substituting (2.67) into (2.66), we obtain

$$
\begin{aligned}
&P(u)\Lambda_3(u)U_m - \partial_t(\Lambda_3(u)U_m) \\
&= -\varepsilon\, M_3(F'', \partial_x u_x)u_x + 5b_1 M_3(\partial_x V_m, \partial_x^2 u_x)u_x + 5b_1 M_3(\partial_x V_m, \partial_x u_x)\partial_x u_x \\
&\quad + d_0 M_3 |u_x|^2(\partial_x V_m, \partial_x u_x)u + d_0'' M_3 |u_x|^2(\partial_x V_m, u_x)u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right).
\end{aligned} \tag{2.68}
$$

We now recall (2.38) and combine (2.49), (2.58) and (2.68) to obtain

$$
\begin{aligned}
&P(u)\Lambda(u)U_m - \partial_t(\Lambda(u)U_m) \\
&= -\varepsilon\left\{M_1|u_x|^2 F' + M_2(F', u_x)u_x + M_3(F'', \partial_x u_x)u_x\right\} \\
&\quad + 10b_1 M_1(\partial_x u_x, u_x)\partial_x^2 V_m + 5b_1 M_2(\partial_x^2 V_m, \partial_x u_x)u_x \\
&\quad + 5b_1 M_2(\partial_x^2 V_m, u_x)\partial_x u_x + 5b_1 M_3(\partial_x V_m, \partial_x^2 u_x)u_x \\
&\quad + \alpha_1 |u_x|^2(\partial_x^2 V_m, u_x)u + \alpha_2(\partial_x^2 u_x, u_x)\partial_x V_m + \alpha_3|\partial_x u_x|^2 \partial_x V_m \\
&\quad + \alpha_4(\partial_x V_m, u_x)(\partial_x u_x, u_x)u + \alpha_5|u_x|^2(\partial_x V_m, \partial_x u_x)u + \alpha_6|u_x|^2(\partial_x V_m, u_x)u_x \\
&\quad + \alpha_7|u_x|^4\partial_x V_m + \alpha_8(\partial_x V_m, \partial_x^2 u_x)u_x + \alpha_9(\partial_x V_m, \partial_x u_x)\partial_x u_x \\
&\quad + \alpha_{10}(\partial_x V_m, u_x)\partial_x^2 u_x + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right),
\end{aligned} \tag{2.69}
$$

where $\alpha_1, \ldots, \alpha_{10}$ are real constants. Although each of the constants may depend on (M_1, M_2, M_3), the exact form is not required except for that $\alpha_8 = \alpha_{10} = 10b_1 M_2$. See Remark2. We are now ready to estimate I_3. Substituting (2.69), we can write

$$
\begin{aligned}
I_3 = \varepsilon &\int_{\mathbb{T}} (\widetilde{F_6}, V_m)dx + 10b_1 M_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)(\partial_x^2 V_m, V_m)dx \\
&+ 5b_1 M_2 \int_{\mathbb{T}} (\partial_x^2 V_m, \partial_x u_x)(V_m, u_x)dx + 5b_1 M_2 \int_{\mathbb{T}} (\partial_x^2 V_m, u_x)(V_m, \partial_x u_x)dx \\
&+ 5b_1 M_3 \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx + \alpha_1 \int_{\mathbb{T}} |u_x|^2(\partial_x^2 V_m, u_x)(u, V_m)dx \\
&+ \int_{\mathbb{T}} (\widetilde{P_{1,1}}(u)\partial_x V_m, V_m)dx + \int_{\mathbb{T}} (\widetilde{P_{1,2}}(u)\partial_x V_m, V_m)dx + \int_{\mathbb{T}} (\widetilde{R_m}, V_m)dx,
\end{aligned} \tag{2.70}
$$

where $\widetilde{R_m} = \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right)$ and

$$\begin{aligned}
\widetilde{F_6} &= -M_1|u_x|^2 F' - M_2(F', u_x)u_x - M_3(F'', \partial_x u_x)u_x \\
&= \mathcal{O}\left(\sum_{j=1}^{4} |\partial_x^j U_m|\right) + \mathcal{O}\left(\sum_{j=0}^{m} |\partial_x^j u_x|\right), \qquad (2.71)
\end{aligned}$$

$$\widetilde{P_{1,1}}(u) = \alpha_4(\cdot, u_x)(\partial_x u_x, u_x)u + \alpha_5|u_x|^2(\cdot, \partial_x u_x)u, \qquad (2.72)$$

$$\begin{aligned}
\widetilde{P_{1,2}}(u) &= \alpha_2(\partial_x^2 u_x, u_x) + \alpha_3|\partial_x u_x|^2 + \alpha_6|u_x|^2(\cdot, u_x)u_x + \alpha_7|u_x|^4 \\
&\quad + \alpha_8(\cdot, \partial_x^2 u_x)u_x + \alpha_9(\cdot, \partial_x u_x)\partial_x u_x + \alpha_{10}(\cdot, u_x)\partial_x^2 u_x. \qquad (2.73)
\end{aligned}$$

Observing (2.71), it is easy to obtain

$$\varepsilon \int_{\mathbb{T}} (\widetilde{F_6}, V_m)dx + \int_{\mathbb{T}} (\widetilde{R_m}, V_m)dx \leqslant \frac{\varepsilon}{4}\|\partial_x^3 V_m\|_{L^2}^2 + C\, N_m(u)^2 \qquad (2.74)$$

in the same way as we obtain (2.24). Using the property (2.27) coming from the constraint condition $|u| = 1$,

$$\begin{aligned}
&\alpha_1 \int_{\mathbb{T}} (|u_x|^2(\partial_x^2 V_m, u_x)u, V_m)dx \\
&\leqslant \alpha_1 \int_{\mathbb{T}} |u_x|^2(\partial_x^2 V_m, u_x)(u, U_m)dx + CN_m(u)^2 \\
&\leqslant -\alpha_1(m+1) \int_{\mathbb{T}} |u_x|^2(\partial_x^2 V_m, u_x)(u_x, \partial_x^{m-1} u_x)dx + CN_m(u)^2 \\
&\leqslant \alpha_1(m+1) \int_{\mathbb{T}} |u_x|^2(\partial_x V_m, u_x)(u_x, \partial_x^{m} u_x)dx + CN_m(u)^2 \\
&\leqslant \alpha_1(m+1) \int_{\mathbb{T}} |u_x|^2(\partial_x V_m, u_x)(u_x, V_m)dx + CN_m(u)^2 \\
&\leqslant CN_m(u)^2. \qquad (2.75)
\end{aligned}$$

In the same way as above, using (2.27) coming from $|u| = 1$, we can obtain

$$\int_{\mathbb{T}} (\widetilde{P_{1,1}}(u)\partial_x V_m, V_m)dx \leqslant C\, N_m(u)^2. \qquad (2.76)$$

Observing (2.73) and $\alpha_8 = \alpha_{10}$, we see $\widetilde{P_{1,2}}$ behaves as a symmetric operator. Hence, as we show, e.g., (2.30), (2.32), we use the integration by parts to obtain

$$\int_{\mathbb{T}} (\widetilde{P_{1,2}}(u)\partial_x V_m, V_m)dx \leqslant C\, N_m(u)^2. \qquad (2.77)$$

Remark 2. The fact $\alpha_8 = \alpha_{10}$ ensures the symmetry of $\widetilde{P_{1,2}}$. Indeed we can show

$$\begin{aligned}
&\alpha_8 \int_{\mathbb{T}} ((\partial_x V_m, \partial_x^2 u_x)u_x, V_m)dx + \alpha_{10} \int_{\mathbb{T}} ((\partial_x V_m, u_x)\partial_x^2 u_x, V_m)dx \\
&\leqslant (\alpha_8 - \alpha_{10}) \int_{\mathbb{T}} ((\partial_x V_m, u_x)\partial_x^2 u_x, V_m)dx + CN_m(u)^2 \\
&= CN_m(u)^2.
\end{aligned}$$

By the integration by parts, we deduce

$$\begin{aligned}
&10b_1 M_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)(\partial_x^2 V_m, V_m)dx \\
&= -10b_1 M_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x V_m|^2 dx - 10b_1 M_1 \int_{\mathbb{T}} (\partial_x^2 u_x, u_x)(\partial_x V_m, V_m)dx \\
&\quad - 10b_1 M_1 \int_{\mathbb{T}} |\partial_x u_x|^2 (\partial_x V_m, V_m)dx \\
&\leqslant -10b_1 M_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x V_m|^2 dx + CN_m(u)^2, \qquad (2.78)
\end{aligned}$$

$$\begin{aligned}
&5b_1 M_2 \int_{\mathbb{T}} (\partial_x^2 V_m, \partial_x u_x)(V_m, u_x)dx + 5b_1 M_2 \int_{\mathbb{T}} (\partial_x^2 V_m, u_x)(V_m, \partial_x u_x)dx \\
&= -5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx \\
&\quad - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(V_m, \partial_x u_x)dx - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, u_x)(\partial_x V_m, \partial_x u_x)dx \\
&\quad - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, u_x)(V_m, \partial_x^2 u_x)dx - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(V_m, \partial_x u_x)dx \\
&\leqslant -5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx \\
&\quad - 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, u_x)(\partial_x V_m, \partial_x u_x)dx + 5b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x^2 u_x)(V_m, u_x)dx \\
&\quad + CN_m(u)^2 \\
&\leqslant -10b_1 M_2 \int_{\mathbb{T}} (\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx + CN_m(u)^2, \qquad (2.79)
\end{aligned}$$

Consequently, substituting (2.74), (2.75), (2.76), (2.77), (2.78), (2.79) into

(2.70), we obtain the following estimate

$$
\begin{aligned}
I_3 \leqslant \frac{\varepsilon}{4}\|\partial_x^3 V_m\|_{L^2}^2 + C\,N_m(u)^2 - 10b_1M_1\int_{\mathbb{T}}(\partial_x u_x, u_x)|\partial_x V_m|^2 dx \\
- 10b_1M_2\int_{\mathbb{T}}(\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx \\
+ 5b_1M_3\int_{\mathbb{T}}(\partial_x^2 V_m, \partial_x^2 u_x)(V_m, u_x)dx. \qquad (2.80)
\end{aligned}
$$

Substituting (2.23), (2.24), (2.37), (2.80) into (2.22), we derive

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|V_m\|_{L^2}^2 \leqslant C\,N_m(u)^2 + (\beta_1 - 10b_1M_1)\int_{\mathbb{T}}(\partial_x u_x, u_x)|\partial_x V_m|^2 dx \\
+ (\beta_2 - 10b_1M_2)\int_{\mathbb{T}}(\partial_x V_m, \partial_x u_x)(\partial_x V_m, u_x)dx \\
+ (\beta_3 + 5b_1M_3)\int_{\mathbb{T}}(\partial_x^2 V_m, \partial_x^2 u_x)(V_m, u_x)dx. \qquad (2.81)
\end{aligned}
$$

Since $b_1 \neq 0$ and the constants $\beta_1, \beta_2, \beta_3$ are independent of (M_1, M_2, M_3), we can set $M_1 = \frac{\beta_1}{10b_1}, M_2 = \frac{\beta_2}{10b_1}, M_3 = -\frac{\beta_3}{5b_1}$ to derive the desired estimate

$$
\frac{1}{2}\frac{d}{dt}\|V_m\|_{L^2}^2 \leqslant C N_m(u)^2. \qquad (2.82)
$$

On the other hand, it is now obvious to obtain the following estimate

$$
\frac{1}{2}\frac{d}{dt}\|u_x\|_{H^{m-1}}^2 \leqslant C N_m(u)^2, \qquad (2.83)
$$

permitting loss of derivatives of order one.

From (2.82) and (2.83), we can conclude that the following estimate

$$
\frac{d}{dt}N_m(u^\varepsilon(t))^2 \leqslant C\,N_m(u^\varepsilon(t))^2 \qquad (2.84)
$$

holds for all $t \in [0, T_\varepsilon^\star]$, where C is a positive constant which is independent of $\varepsilon \in (0, 1]$.

Once we obtain the estimate (2.84), it is straightforward to show the uniform boundedness of $N_m(u^\varepsilon)$. Indeed, it follows from (2.84) that

$$
N_m(u^\varepsilon(t))^2 \leqslant N_m(u_0)^2 \exp(Ct), \quad t \in [0, T_\varepsilon^\star]. \qquad (2.85)
$$

Then, by the definition of $T_\varepsilon^\star$, we have

$$2N_8(u_0)^2 = N_8(u^\varepsilon(T_\varepsilon^\star))^2 \leqslant N_8(u_0)^2 \exp(CT_\varepsilon^\star).$$

If $N_m(u_0) \neq 0$, then this shows $2 \leqslant \exp(CT_\varepsilon^\star)$, that is, $(\log 2)/C \leqslant T_\varepsilon^\star$. On the other hand, if $N_m(u_0) = 0$, then $u_{0x} = 0$ and u_0 is a constant map. Then, the constant map $u^\varepsilon \equiv u_0$ turns out to be the unique solution to (2.1)-(2.2), which exists globally in time. This implies $N_m(u^\varepsilon(t)) = 0$ on $[0,\infty)$. In any case, if we set $T = (\log 2)/C$, then we see $0 < T \leqslant T_\varepsilon^\star$ and $\{N_m(u^\varepsilon)\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;\mathbb{R})$. As we observed above (just below (2.10)), this shows that $\{u_x^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^{N+1}))$, which is the desired result. $\square$

3. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $m \geqslant 8$ be a fixed integer. It suffices to solve the problem in the positive direction in time. From Proposition 2.1 and Proposition 2.4, it follows that there exists a constant $T = T(\|u_{0x}\|_{H^8}) > 0$ which is independent of $\varepsilon \in (0,1]$ and exists a family $\{u^\varepsilon\}_{\varepsilon\in(0,1]}$ solving (2.1)-(2.2) on $[0,T]$ such that $\{u_x^\varepsilon\}_{\varepsilon\in(0,1]}$ is bounded in $L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^{N+1}))$. Therefore, by the standard compactness argument, up to a subsequence we can pass to the limit $\varepsilon \to 0$ in the equation (2.1) to find a a map $u \in C([0,T]\times\mathbb{T};\mathbb{S}^N)$ solving (1.1)-(1.2) such that $u_x \in L^\infty(0,T;H^m(\mathbb{T};\mathbb{R}^{N+1})) \cap C([0,T];H^{m-1}(\mathbb{T};\mathbb{R}^{N+1}))$. This completes the proof of the existence of a solution locally in time.

Next, we shall show the uniqueness of the solution u constructed above. Let u and v be solutions to (1.1)-(1.2) satisfying $u_x, v_x \in L^\infty(0,T;H^8(\mathbb{T};\mathbb{R}^{N+1})) \cap C([0,T];H^7(\mathbb{T};\mathbb{R}^{N+1}))$. Set $z = u - v$. To show the uniqueness, it suffices to show $z = 0$. For this purpose, we consider the estimate for

$$D(z(t)) = \left\{\|z(t)\|_{L^2}^2 + \|z_x(t)\|_{L^2}^2 + \|z_{xx}(t)\|_{L^2}^2 + \|\widetilde{W}(t)\|_{L^2}^2\right\}^{1/2}, \tag{3.1}$$
$$\widetilde{W} = z_{xxx} - M_1|u_x|^2 z_x - M_2(z_x,u_x)u_x - M_3(z,\partial_x u_x)u_x,$$

where $M_1, M_2, M_3 \in \mathbb{R}$ are constants which will be taken later. It is easy to see that there exists a constant $C > 1$ depending on $\|u_x\|_{C([0,T];H^3)}$, $\|v_x\|_{C([0,T];H^3)}$ and on (M_1,M_2,M_3) such that

$$\frac{1}{C}\|z(t)\|_{H^3} \leqslant D(z(t)) \leqslant C\|z(t)\|_{H^3} \tag{3.2}$$

for all $t \in [0,T]$. Noting this, we shall show that there exists a positive constant C depending on $\|u_x\|_{C([0,T];H^7)}$, $\|v_x\|_{C([0,T];H^7)}$ and on (M_1, M_2, M_3) such that

$$\frac{1}{2}\frac{d}{dt}D(z(t))^2 \leqslant C\, D(z(t))^2 \tag{3.3}$$

for all $t \in [0,T]$. If it is true, then we have $0 \leqslant D(z(t)) \leqslant D(z(0))e^{2Ct}$. Then we find $D(z(t)) = 0$ on $[0,T]$, since $D(z(0)) = 0$. This shows $z = 0$.

To show (3.3), we set $U := u_{xxx}$, $V := v_{xxx}$, and $W := z_{xxx} = U - V$. From (1.1), it follows that $\partial_t U = \partial_x^3 u_t = b_1\partial_x^5 U + 5b_1(\partial_x^4 U, u_x)u + \cdots$. The same PDE as above is satisfied by V. After lengthy calculations taking the difference of the equation for U and that for V, we obtain

$$\partial_t W = P(u,v)W + R.$$

Here $R = \mathcal{O}(|z| + |z_x| + |z_{xx}| + |W|)$ and

$$P(u,v) = b_1\partial_x^5 + P_4(u)\partial_x^4 + P_3(u)\partial_x^3 + P_2(u)\partial_x^2 + P_1(u,v)\partial_x, \tag{3.4}$$
$$P_1(u,v) = P_1(u) + d_{19}(\cdot, \partial_x^3 v_x)u, \tag{3.5}$$

where d_{19} is a real constant, and $P_4(u)$, $P_3(u)$, $P_2(u)$, $P_1(u)$ are defined by (2.17), (2.18), (2.19), (2.20) respectively. In addition, we can also write $P(u,v) = P(u) + d_{19}(\cdot, \partial_x^3 v_x)u$ where $P(u)$ is defined by (2.16). (Though the coefficients $d_0, \dots, d_{18}$ may not coincide, the difference is not essential.)

Remark 3. The reason why the second term of the right hand side of (3.5) appears depending on v comes from the following: Observing $d_{13}(\cdot, \partial_x^3 u_x)u$ in (2.20), the term of the form $(\partial_x U, \partial_x U)u$ is included in the right hand side of $\partial_t U = b_1\partial_x^5 U + 5b_1(\partial_x U, u_x)u + \cdots$. Thus, we see $\partial_t W$ includes the term of the form

$$\begin{aligned}(\partial_x U, \partial_x U)u - (\partial_x V, \partial_x V)v &= (\partial_x W, \partial_x U)u + (\partial_x V, \partial_x W)u + (\partial_x V, \partial_x V)z \\ &= (\partial_x W, \partial_x^3 u_x)u + (\partial_x W, \partial_x^3 v_x)u + \mathcal{O}(|z|).\end{aligned}$$

Note that $\widetilde{W} = W - \Lambda(u)W$ where $\Lambda(u) = \Lambda_1(u) + \Lambda_2(u) + \Lambda_3(u)$ is just as same as that defined by (2.6) and (2.7). Using the expression, we have

$$\begin{aligned}\partial_t \widetilde{W} &= \partial_t W - \partial_t(\Lambda(u)W) = P(u,v)W - \partial_t(\Lambda(u)W) + R \\ &= P(u,v)\widetilde{W} + P(u,v)\Lambda(u)W - \partial_t(\Lambda(u)W) + R. \end{aligned}\tag{3.6}$$

Since $m \geqslant 8$, we see $u_x, v_x \in C([0,T]; H^7(\mathbb{T};\mathbb{R}^{N+1}))$, $\partial_t \widetilde{W} = b_1 \partial_x^5 \widetilde{W} + \cdots \in C([0,T]; L^2(\mathbb{T};\mathbb{R}^{N+1}))$, and thus $\widetilde{W} \in C^1([0,T]; L^2(\mathbb{T};\mathbb{R}^{N+1}))$. Using (3.6) and $P(u,v) = P(u) + d_{19}(\cdot, \partial_x^3 v_x)u$, we have

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{W}\|_{L^2}^2 = \int_{\mathbb{T}} (\partial_t \widetilde{W}, \widetilde{W})dx =: I_1 + I_2 + I_3 + I_4, \tag{3.7}$$

where

$$I_1 = \int_{\mathbb{T}} (P(u)\widetilde{W}, \widetilde{W})dx, \quad I_2 = d_{19}\int_{\mathbb{T}} (\partial_x \widetilde{W}, \partial_x^3 v_x)(u, \widetilde{W})dx$$
$$I_3 = \int_{\mathbb{T}} (P(u,v)\Lambda(u)W - \partial_t(\Lambda(u)W), \widetilde{W})dx, \quad I_4 = \int_{\mathbb{T}} (R, \widetilde{W})dx.$$

We estimate $I_1, \ldots, I_4$ separately. In the computation, we often use the property $\widetilde{W} = W + \mathcal{O}(|z| + |z_x|)$ and (3.2) without any comments. In addition, here and hereafter, positive constants depending on $\|u_x\|_{C([0,T];H^7)}$ and $\|v_x\|_{C([0,T];H^7)}$ are denoted by the same C.

First, since $R = \mathcal{O}(|z| + |z_x| + |z_{xx}| + |W|)$, we easily obtain $\|R\|_{L^2} \leqslant C\|z\|_{H^3} \leqslant CD(z)$. This shows

$$I_4 \leqslant \|\widetilde{R}\|_{L^2}\|\widetilde{W}\|_{L^2} \leqslant C\,D(z)^2. \tag{3.8}$$

Second, since $(u, z_x) = (u, u_x - v_x) = -(u, v_x) = -(u - v, v_x) = -(z, v_x)$ follows from $|u|^2 = |v|^2 = 1$, we deduce

$$\begin{aligned}
&(\widetilde{W}, u)\\
&= (z_{xxx}, u) - M_1|u_x|^2(z_x, u) - M_2(z_x, u_x)(u_x, u) - M_3(z, u_{xx})(u_x, u)\\
&= \partial_x^2\{(z_x, u)\} - 2(z_{xx}, u_x) - (z_x, u_{xx}) - M_1|u_x|^2(z_x, u)\\
&= -\partial_x^2\{(z, v_x)\} - 2(z_{xx}, u_x) - (z_x, u_{xx}) - M_1|u_x|^2(z_x, u),
\end{aligned} \tag{3.9}$$

which yields $(\widetilde{W}, u) = \mathcal{O}(|z_{xx}| + |z_x| + |z|)$. Using the integration by parts and (3.9), we have

$$\begin{aligned}
I_2 &= -d_{19}\int_{\mathbb{T}} (\widetilde{W}, \partial_x^4 v_x)(u, \widetilde{W}) - d_{19}\int_{\mathbb{T}} (\widetilde{W}, \partial_x^3 v_x)\partial_x\{(u, \widetilde{W})\}dx\\
&\leqslant C\|\widetilde{W}\|_{L^2}^2 + C\|\widetilde{W}\|_{L^2}\|\mathcal{O}(|W| + |z_{xx}| + |z_x| + |z|)\|_{L^2}\\
&\leqslant C\,D(z)^2.
\end{aligned} \tag{3.10}$$

Third, by the almost same lengthy argument as we show (2.37), we can deduce

$$
\begin{aligned}
I_1 \leqslant C\,D(z)^2 &+ \beta_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x \widetilde{W}|^2 dx \\
&+ \beta_2 \int_{\mathbb{T}} (\partial_x \widetilde{W}, \partial_x u_x)(\partial_x \widetilde{W}, u_x) dx + \beta_3 \int_{\mathbb{T}} (\partial_x \widetilde{W}, \partial_x^2 u_x)(\widetilde{W}, u_x) dx, \quad (3.11)
\end{aligned}
$$

where $\beta_1, \beta_2, \beta_3$ are real constants but are independent of (M_1, M_2, M_3). This may be reasonable because $P_4(u), P_3(u), P_2(u), P_1(u)$ here are respectively the same as that given by (2.17), (2.18) , (2.19), (2.20) up to the coefficients. However, a few remarks are in order on the difference of the argument here and that to obtain (2.37): To obtain (2.37), the property (2.27) works effectively to estimate the terms containing (U_m, u). On the other hand, to obtain (3.11) here, we cannot use (2.27) directly, because what we need to estimate is not (U_m, u) but $(\widetilde{W}, u)$. Despite that, in the same way as we estimate I_2 above, the property (3.9) works (instead of (2.27)) effectively to obtain (3.11). Hence, the difference is not essential.

Fourth, by the essentially same computation as we obtain (2.47), (2.56), (2.66) with $\varepsilon = 0$, we can deduce

$$
\begin{aligned}
I_3 &\leqslant \int_{\mathbb{T}} (P(u)\Lambda(u)W - \partial_t(\Lambda(u)W), \widetilde{W}) dx + C\,D(z)^2 \\
&\leqslant b_1 \int_{\mathbb{T}} ([\partial_x^5, \Lambda_1(u) + \Lambda_2(u) + \Lambda_3(u)]W, \widetilde{W}) dx + C\,D(z)^2. \quad (3.12)
\end{aligned}
$$

Here, by the same computation as we obtain (2.48) , (2.57) , (2.67) and $W = \widetilde{W} + \mathcal{O}(|z| + |z_x|)$, we obtain

$$
\begin{aligned}
\left[\partial_x^5, \Lambda_1(u)\right] W = 10M_1(\partial_x u_x, u_x)\partial_x^2 \widetilde{W} &+ 20M_1(\partial_x^2 u_x, u_x)\partial_x \widetilde{W} \\
&+ 20M_1 |\partial_x u_x|^2 \partial_x \widetilde{W} + \mathcal{O}(|z| + |z_x| + |z_{xx}| + |W|),
\end{aligned}
$$

$$
\begin{aligned}
&\left[\partial_x^5, \Lambda_2(u)\right] W \\
&= 5M_2(\partial_x^2 \widetilde{W}, \partial_x u_x) u_x + 5M_2(\partial_x^2 \widetilde{W}, u_x)\partial_x u_x + 10M_2(\partial_x \widetilde{W}, \partial_x^2 u_x) u_x \\
&\quad + 20M_2(\partial_x \widetilde{W}, \partial_x u_x)\partial_x u_x + 10M_2(\partial_x \widetilde{W}, u_x)\partial_x^2 u_x \\
&\quad + \mathcal{O}(|z| + |z_x| + |z_{xx}| + |W|),
\end{aligned}
$$

$$
\begin{aligned}
&\left[\partial_x^5, \Lambda_3(u)\right] W \\
&= 5M_3(\partial_x \widetilde{W}, \partial_x^2 u_x) u_x + 5M_3(\partial_x \widetilde{W}, \partial_x u_x)\partial_x u_x + \mathcal{O}(|z| + |z_x| + |z_{xx}| + |W|).
\end{aligned}
$$

Substituting them into (3.12) and using the integration by parts, we obtain

$$
\begin{aligned}
I_3 &\leqslant -10b_1M_1\int_{\mathbb{T}}(\partial_x u_x, u_x)|\partial_x\widetilde{W}|^2dx - 10b_1M_2\int_{\mathbb{T}}(\partial_x\widetilde{W}, \partial_x u_x)(\partial_x\widetilde{W}, u_x)dx \\
&\quad + 5b_1M_3\int_{\mathbb{T}}(\partial_x\widetilde{W}, \partial_x^2 u_x)(\widetilde{W}, u_x)dx + CD(z)^2. \qquad (3.13)
\end{aligned}
$$

We omit the detail, since the computation is almost same as that we obtain (2.80).

Consequently, substituting (3.8), (3.10), (3.11), (3.13) into (3.7), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\widetilde{W}\|_{L^2}^2 \leqslant C\,D(z)^2 &+ (\beta_1 - 10b_1M_1)\int_{\mathbb{T}}(\partial_x u_x, u_x)|\partial_x\widetilde{W}|^2dx \\
&+ (\beta_2 - 10b_1M_2)\int_{\mathbb{T}}(\partial_x\widetilde{W}, \partial_x u_x)(\partial_x\widetilde{W}, u_x)dx \\
&+ (\beta_3 + 5b_1M_3)\int_{\mathbb{T}}(\partial_x\widetilde{W}, \partial_x^2 u_x)(\widetilde{W}, u_x)dx.
\end{aligned}
$$

Since $b_1 \neq 0$ and since the constants $\beta_1, \beta_2, \beta_3$ are independent of (M_1, M_2, M_3), we can set $M_1 = \frac{\beta_1}{10b_1}$, $M_2 = \frac{\beta_2}{10b_1}$, $M_3 = -\frac{\beta_3}{5b_1}$ to conclude that there exists a constant $C > 0$ depending on $\|u_x\|_{C([0,T];H^7)}$ and on $\|v_x\|_{C([0,T];H^7)}) > 0$ such that

$$
\frac{1}{2}\frac{d}{dt}\|\widetilde{W}(t)\|_{L^2}^2 \leqslant CD(z(t))^2 \qquad (3.14)
$$

for all $t \in [0, T]$. It is now easy to obtain the estimate of the form

$$
\frac{1}{2}\frac{d}{dt}\left\{\|z(t)\|_{L^2}^2 + \|z_x(t)\|_{L^2}^2\right\} \leqslant C\,D(z(t))^2 \qquad (3.15)
$$

permitting loss of derivatives of order one. From (3.14) and (3.15), we derive (3.3), which is the desired result to conclude the uniqueness of the solution u.

Once the uniqueness is established, we can prove the time-continuity of $\partial_x^m u_x$ in L^2 by the argument following [13] (see e.g., [6, 21] for more details.), which implies $u_x \in C([0,T]; H^m(\mathbb{T}; \mathbb{R}^{N+1}))$. This completes the proof of Theorem 1.1. □

4. Appendix: Proof of Proposition 2.3

Proof of Proposition 2.3. we define a function $h = h(t,x) : [0,T_\varepsilon] \times \mathbb{T} \to \mathbb{R}$ by $h(t,x) = |u(t,x)|^2 - 1$. It suffices to show $h = 0$. A simple computation yields

$$\partial_x h = 2(u, u_x), \tag{4.1}$$
$$\partial_x^2 h = 2\left\{(u, \partial_x u_x) + |u_x|^2\right\}, \tag{4.2}$$
$$\partial_x^3 h = 2\left\{(u, \partial_x^2 u_x) + 3(u_x, \partial_x u_x)\right\}, \tag{4.3}$$
$$\partial_x^4 h = 2\left\{(u, \partial_x^3 u_x) + 4(u_x, \partial_x^2 u_x) + 3|\partial_x u_x|^2\right\}, \tag{4.4}$$
$$\partial_x^5 h = 2\left\{(u, \partial_x^4 u_x) + 5(u_x, \partial_x^3 u_x) + 10(\partial_x u_x, \partial_x^2 u_x)\right\}, \tag{4.5}$$
$$\partial_x^6 h = 2\left\{(u, \partial_x^5 u_x) + 6(u_x, \partial_x^4 u_x) + 15(\partial_x u_x, \partial_x^3 u_x) + 10|\partial_x^2 u_x|^2\right\}. \tag{4.6}$$

Since u satisfies (2.1),

$$\frac{1}{2}\partial_t h = (u, u_t) = \varepsilon\,(u, F_6) + (u, F_5), \tag{4.7}$$

where F_6 and F_5 have been defined by (2.3) and (2.4) respectively. From (2.3) and (4.6), it follows that

$$\begin{aligned}(u, F_6) &= (u, \partial_x^5 u_x) + 6(\partial_x^4 u_x, u_x)|u|^2 + 15(\partial_x^3 u_x, \partial_x u_x)|u|^2 + 10|\partial_x^2 u_x|^2|u|^2 \\ &= \frac{1}{2}\partial_x^6 h + \left\{6(\partial_x^4 u_x, u_x) + 15(\partial_x^3 u_x, \partial_x u_x) + 10|\partial_x^2 u_x|^2\right\} h.\end{aligned} \tag{4.8}$$

In the same way, from (4.1), (4.2), (4.3), and (4.5), it follows that

$$
\begin{aligned}
(u, F_5) &= \left\{ \frac{1}{2} b_1 \partial_x^5 h - 5b_1 (u_x, \partial_x^3 u_x) - 10 b_1 (\partial_x u_x, \partial_x^2 u_x) \right\} + 5b_1 (\partial_x^3 u_x, u_x) |u|^2 \\
&\quad + 10 b_1 (\partial_x^2 u_x, \partial_x u_x) |u|^2 + \left\{ \frac{1}{2} b_2 |u_x|^2 \partial_x^3 h - 3b_2 |u_x|^2 (u_x, \partial_x u_x) \right\} \\
&\quad + \frac{1}{2} b_3 (\partial_x^2 u_x, u_x) \partial_x h + \left\{ \frac{1}{2} b_4 (\partial_x u_x, u_x) \partial_x^2 h - b_4 (\partial_x u_x, u_x) |u_x|^2 \right\} \\
&\quad + \frac{1}{2} b_5 |\partial_x u_x|^2 \partial_x h + b_6 |u_x|^2 (\partial_x u_x, u_x) |u|^2 + \frac{1}{2} b_7 |u_x|^4 \partial_x h \\
&= \frac{1}{2} b_1 \partial_x^5 h + \frac{1}{2} b_2 |u_x|^2 \partial_x^3 h + \frac{1}{2} b_4 (\partial_x u_x, u_x) \partial_x^2 h \\
&\quad + \left\{ \frac{1}{2} b_3 (\partial_x^2 u_x, u_x) + \frac{1}{2} b_5 |\partial_x u_x|^2 + \frac{1}{2} b_7 |u_x|^4 \right\} \partial_x h \\
&\quad + \{ 5b_1 (\partial_x^3 u_x, u_x) + 10 b_1 (\partial_x^2 u_x, \partial_x u_x) + b_6 |u_x|^2 (\partial_x u_x, u_x) \} h \\
&\quad + (5b_1 - 5b_1)(\partial_x^3 u_x, u_x) + (10 b_1 - 10 b_1)(\partial_x^2 u_x, \partial_x u_x) \\
&\quad + (b_6 - 3b_2 - b_4) |u_x^2| (\partial_x u_x, u_x).
\end{aligned}
$$

By the assumption on the coefficients b_i, $i = 1, 2, \ldots, 7$, the last three terms of the right hand side of above vanish. That is, we have

$$
\begin{aligned}
(u, F_5) &= \frac{1}{2} b_1 \partial_x^5 h + \frac{1}{2} b_2 |u_x|^2 \partial_x^3 h + \frac{1}{2} b_4 (\partial_x u_x, u_x) \partial_x^2 h \\
&\quad + \left\{ \frac{1}{2} b_3 (\partial_x^2 u_x, u_x) + \frac{1}{2} b_5 |\partial_x u_x|^2 + \frac{1}{2} b_7 |u_x|^4 \right\} \partial_x h \\
&\quad + \{ 5b_1 (\partial_x^3 u_x, u_x) + 10 b_1 (\partial_x^2 u_x, \partial_x u_x) + b_6 |u_x|^2 (\partial_x u_x, u_x) \} h. \qquad (4.9)
\end{aligned}
$$

Substituting (4.8) and (4.9) into (4.7), we obtain

$$
\begin{aligned}
\partial_t h &= \varepsilon \left[\partial_x^6 h + \left\{ 12 (\partial_x^4 u_x, u_x) + 30 (\partial_x^3 u_x, \partial_x u_x) + 20 |\partial_x^2 u_x|^2 \right\} h \right] \\
&\quad + b_1 \partial_x^5 h + b_2 |u_x|^2 \partial_x^3 h + b_4 (\partial_x u_x, u_x) \partial_x^2 h \\
&\quad + \{ b_3 (\partial_x^2 u_x, u_x) + b_5 |\partial_x u_x|^2 + b_7 |u_x|^4 \} \partial_x h \\
&\quad + \{ 10 b_1 (\partial_x^3 u_x, u_x) + 20 b_1 (\partial_x^2 u_x, \partial_x u_x) + 2 b_6 |u_x|^2 (\partial_x u_x, u_x) \} h. \qquad (4.10)
\end{aligned}
$$

Remark 4. By the form of F_5 and the choice of the form of F_6, the right hand side of (4.10) can be written by a linear combination of $h, \partial_x h, \ldots, \partial_x^6 h$. This will be helpful to obtain the energy estimate for h below.

We next introduce

$$\widetilde{G} = \partial_x^2 h - M_1 |u_x|^2 \, h, \tag{4.11}$$

$$E(h(t)) = \left\{ \|h(t)\|_{L^2}^2 + \|\partial_x h(t)\|_{L^2}^2 + \|\widetilde{G}(t)\|_{L^2}^2 \right\}^{1/2}, \tag{4.12}$$

where $M_1 \in \mathbb{R}$ is a constant which will be taken later. There exists a constant $C > 1$ depending on $\|u_x\|_{C([0,T_\varepsilon];H^1)}$ and on M_1 such that

$$\frac{1}{C}\|h(t)\|_{H^2} \leqslant E(h(t)) \leqslant C\|h(t)\|_{H^2} \tag{4.13}$$

for all $t \in [0, T_\varepsilon]$. We shall show that there exists a constant $C > 0$ depending on $\|u_x\|_{C([0,T_\varepsilon];H^7)}$ (and on M_1) such that

$$\frac{1}{2}\frac{d}{dt}E(h(t))^2 \leqslant C\, E(h(t))^2 \tag{4.14}$$

for all $t \in [0, T_\varepsilon]$. If it is true, then we have $0 \leqslant E(h(t)) \leqslant E(h(0))e^{2Ct} = 0$, which implies $h = 0$. ($E(h(0)) = 0$ holds since $|u_0(x)|^2 = 1$ for all $x \in \mathbb{T}$.)

For this purpose, we set $G = \partial_x^2 h$. Then we can write $\widetilde{G} = G - \Lambda(u)G$, where $\Lambda(u) = M_1|u_x|^2\partial_x^{-2}$ which is just $\Lambda_1(u)$ defined by (2.6)-(2.7) in Section 2. After lengthy calculations using (4.10) and the Leibniz rule, we obtain

$$\partial_t G (= \partial_x^2 \partial_t h) = \varepsilon F + P(u)G + R, \tag{4.15}$$

where

$$F = \partial_x^8 h + \mathcal{O}\left(|\partial_x^6 u_x| + |\partial_x^5 u_x| \cdots + |u_x|\right)(\partial_x^2 h + \partial_x h + h), \tag{4.16}$$

$$P(u) = b_1\partial_x^5 + P_3(u)\partial_x^3 + P_2(u)\partial_x^2 + P_1(u)\partial_x,$$

$$P_3(u) = d_0|u_x|^2, \tag{4.17}$$

$$P_2(u) = d_1(\partial_x u_x, u_x), \tag{4.18}$$

$$P_1(u) = d_2(\partial_x^2 u_x, u_x) + d_3|\partial_x u_x|^2 + d_4|u_x|^4, \tag{4.19}$$

$$R = \mathcal{O}\left(|\partial_x^5 u_x| + |\partial_x^4 u_x| \cdots + |u_x|\right)(\partial_x^2 h + \partial_x h + h), \tag{4.20}$$

and each of $d_0, \ldots, d_4$ is a real constant which is given by a linear combination of $b_1, \ldots, b_7$, the exact form of which will not be required. By (4.15) and $\widetilde{G} = G - \Lambda(u)G$, we have

$$\begin{aligned}\partial_t \widetilde{G} &= \varepsilon F + P(u)G + R - \partial_t(\Lambda(u)G) \\ &= \varepsilon F + P(u)\widetilde{G} + \{P(u)\Lambda(u)G - \partial_t(\Lambda(u)G)\} + R.\end{aligned} \tag{4.21}$$

We use (4.21) to see

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\widetilde{G}\|_{L^2}^2 &= \int_{\mathbb{T}} (\partial_t \widetilde{G}, \widetilde{G})dx =: \varepsilon\, J_1 + J_2 + J_3 + J_4, \qquad (4.22)\\
J_1 &= \int_{\mathbb{T}} (F, \widetilde{G})dx, \quad J_2 = \int_{\mathbb{T}} (P(u)\widetilde{G}, \widetilde{G})dx,\\
J_3 &= \int_{\mathbb{T}} (P(u)\Lambda(u)G - \partial_t(\Lambda(u)G), \widetilde{G})dx, \quad J_4 = \int_{\mathbb{T}} (R, \widetilde{G})dx.
\end{aligned}
$$

We estimate J_4, εJ_1, J_2, J_3 separately. For J_4, from (4.20) we see $\|R\|_{L^2} \leqslant C\,\|h\|_{H^2}$ where C is a positive constant depending on $\|u_x\|_{C([0,T_\varepsilon];H^6)}$. Recalling (4.13), it is easy to deduce

$$
J_4 \leqslant \|\widetilde{R}\|_{L_2}\|\widetilde{G}\|_{L_2} \leqslant C(\|u_x\|_{C([0,T_\varepsilon];H^6)})E(h(t))^2. \qquad (4.23)
$$

For εJ_1, from (4.16) and the definition of $\widetilde{G}$, it follows that

$$
\begin{aligned}
F &= \partial_x^6\{\widetilde{G} + M_1|u_x|^2 h\} + \mathcal{O}\left(|\partial_x^6 u_x| + |\partial_x^5 u_x| \cdots + |u_x|\right)(\partial_x^2 h + \partial_x h + h)\\
&= \partial_x^6 \widetilde{G} + \mathcal{O}\left(|\partial_x^6 u_x| + |\partial_x^5 u_x| \cdots + |u_x|\right)(\partial_x^6 h + \partial_x^5 h + \cdots + h).
\end{aligned}
$$

Hence, in the same way as we obtain (2.24), we obtain

$$
\varepsilon\, J_1 \leqslant -\frac{\varepsilon}{2}\,\|\partial_x^3 \widetilde{G}\|_{L^2}^2 + C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h)^2. \qquad (4.24)
$$

For J_2, we demonstrate the computation in a little more detail to see $\Lambda_2(u)$ and $\Lambda_3(u)$ are not required when we define $\widetilde{G}$. To begin with, by using integration by parts repeatedly, we deduce

$$
b_1 \int_{\mathbb{T}} (\partial_x^5 \widetilde{G}, \widetilde{G})dx = b_1 \int_{\mathbb{T}} (\partial_x^3 \widetilde{G}, \partial_x^2 \widetilde{G})dx = 0, \qquad (4.25)
$$

$$
\begin{aligned}
&\int_{\mathbb{T}} (P_3(u)\partial_x^3 \widetilde{G}, \widetilde{G})dx = d_0 \int_{\mathbb{T}} |u_x|^2 (\partial_x^3 \widetilde{G}, \widetilde{G})dx\\
&= -d_0 \int_{\mathbb{T}} |u_x|^2 (\partial_x^2 \widetilde{G}, \partial_x \widetilde{G})dx - 2d_0 \int_{\mathbb{T}} (\partial_x u_x, u_x)(\partial_x^2 \widetilde{G}, \widetilde{G})dx\\
&= d_0 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x \widetilde{G}|^2 dx - 2d_0 \int_{\mathbb{T}} (\partial_x u_x, u_x)(\partial_x^2 \widetilde{G}, \widetilde{G})dx.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\int_{\mathbb{T}}(P_3(u)\partial_x^3\widetilde{G},\widetilde{G})dx+\int_{\mathbb{T}}(P_2(u)\partial_x^2\widetilde{G},\widetilde{G})dx\\
&=d_0\int_{\mathbb{T}}(\partial_x u_x,u_x)|\partial_x\widetilde{G}|^2dx+(d_1-2d_0)\int_{\mathbb{T}}(\partial_x u_x,u_x)(\partial_x^2\widetilde{G},\widetilde{G})dx\\
&=d_0\int_{\mathbb{T}}(\partial_x u_x,u_x)|\partial_x\widetilde{G}|^2dx-(d_1-2d_0)\int_{\mathbb{T}}(\partial_x u_x,u_x)|\partial_x\widetilde{G}|^2dx\\
&\quad-(d_1-2d_0)\int_{\mathbb{T}}\partial_x\{(\partial_x u_x,u_x)\}(\partial_x\widetilde{G},\widetilde{G})dx\\
&=(3d_0-d_1)\int_{\mathbb{T}}(\partial_x u_x,u_x)|\partial_x\widetilde{G}|^2dx-(d_1-2d_0)\int_{\mathbb{T}}\partial_x\{(\partial_x u_x,u_x)\}(\partial_x\widetilde{G},\widetilde{G})dx.
\end{aligned}
$$

Furthermore, by integration by parts and the Sobolev embedding $H^1(\mathbb{T};\mathbb{R}^{N+1})\subset C(\mathbb{T};\mathbb{R}^{N+1})$, we have

$$
\begin{aligned}
\int_{\mathbb{T}}\partial_x\{(\partial_x u_x,u_x)\}(\partial_x\widetilde{G},\widetilde{G})dx&=-\frac{1}{2}\int_{\mathbb{T}}\partial_x^2\{(\partial_x u_x,u_x)\}|\widetilde{G}|^2dx\\
&\leqslant C(\|u_x\|_{C([0,T_\varepsilon];H^4)})\|\widetilde{G}\|_{L^2}^2\\
&\leqslant C(\|u_x\|_{C([0,T_\varepsilon];H^4)})\,E(h(t))^2.
\end{aligned}
$$

By combining these estimates, we obtain

$$
\begin{aligned}
&\int_{\mathbb{T}}(P_3(u)\partial_x^3\widetilde{G},\widetilde{G})dx+\int_{\mathbb{T}}(P_2(u)\partial_x^2\widetilde{G},\widetilde{G})dx\\
&\leqslant(3d_0-d_1)\int_{\mathbb{T}}(\partial_x u_x,u_x)|\partial_x\widetilde{G}|^2dx+C(\|u_x\|_{C([0,T_\varepsilon];H^4)}\|)\,E(h(t))^2.
\end{aligned}
\tag{4.26}
$$

Noting $P_1(u)$ behaves as a symmetric operator, we use the integration by parts to deduce

$$
\begin{aligned}
\int_{\mathbb{T}}(P_1(u)\partial_x\widetilde{G},\widetilde{G})dx&=-\frac{1}{2}\int_{\mathbb{T}}\partial_x\{P_1(u)\}|\widetilde{G}|^2dx\\
&\leqslant C(\|u_x\|_{C([0,T_\varepsilon];H^4)})E(h(t))^2.
\end{aligned}
\tag{4.27}
$$

Combining (4.25), (4.26), and (4.27), we obtain

$$
J_2\leqslant(3d_0-d_1)\int_{\mathbb{T}}(\partial_x u_x,u_x)|\partial_x\widetilde{G}|^2dx+C(\|u_x\|_{C([0,T_\varepsilon];H^4)})\,E(h(t))^2. \tag{4.28}
$$

For J_3, note first that $u_x \in C([0,T_\varepsilon]; H^7(\mathbb{T};\mathbb{R}^{N+1}))$ and thus $u_t = \varepsilon\, \partial_x^5 u_x + \cdots \in C([0,T_\varepsilon]; H^2(\mathbb{T};\mathbb{R}^{N+1}))$ follow from $m \geq 7$. Hence we can see

$$\partial_t(\Lambda(u)G) = \partial_t(M_1|u_x|^2 h) = M_1|u_x|^2\partial_t h + \mathcal{O}(\|u_x\|_{C([0,T_\varepsilon];H^7)}|h|).$$

Therefore, by the almost same computation as we obtain (2.47), we can deduce

$$J_3 \leqslant \varepsilon \int_{\mathbb{T}} (\widetilde{F}, \widetilde{G})dx + b_1 \int_{\mathbb{T}} ([\partial_x^5, \Lambda(u)]G, \widetilde{G})dx + C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h)^2,$$

where $\widetilde{F} = \mathcal{O}(|\partial_x^6 u_x| + |\partial_x^5 u_x| + \cdots + |u_x|)(\partial_x^6 h + \partial_x^5 h + \cdots + h)$. In the same way as we estimate εJ_1 using the Gagliardo-Nirenberg inequality, the Young inequality, and $0 < \varepsilon \leqslant 1$, we obtain

$$\varepsilon \int_{\mathbb{T}} (\widetilde{F}, \widetilde{G})dx \leqslant \frac{\varepsilon}{4}\|\partial_x^3 \widetilde{G}\|_{L^2}^2 + C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h)^2. \tag{4.29}$$

Recall that $\Lambda(u)$ here is just $\Lambda_1(u)$ defined by (2.6)-(2.7) in Section 2. Hence, by the same computation as we obtain (2.48) and $G = \widetilde{G} + \mathcal{O}(|h|)$, we obtain

$$\begin{aligned}\left[\partial_x^5, \Lambda(u)\right] G = 10M_1(\partial_x u_x, u_x)\partial_x^2 \widetilde{G} + 20M_1(\partial_x^2 u_x, u_x)\partial_x \widetilde{G} \\ + 20M_1|\partial_x u_x|^2 \partial_x \widetilde{G} + \mathcal{O}(|\partial_x^2 h| + |\partial_x h| + |h|).\end{aligned} \tag{4.30}$$

Substituting (4.30) and using the integration by parts, we deduce

$$\begin{aligned}&b_1 \int_{\mathbb{T}} ([\partial_x^5, \Lambda(u)]G, \widetilde{G})dx \\ &\leqslant C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h)^2 - 10b_1 M_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x \widetilde{G}|^2\, dx.\end{aligned} \tag{4.31}$$

We omit the detail, since the computation is almost same as that we obtain (2.80). Combining (4.29) and (4.31), we obtain

$$J_3 \leqslant \frac{\varepsilon}{4}\|\partial_x^3 \widetilde{G}\|_{L^2}^2 + C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h)^2 - 10b_1 M_1 \int_{\mathbb{T}} (\partial_x u_x, u_x)|\partial_x \widetilde{G}|^2\, dx. \tag{4.32}$$

Collecting (4.23), (4.24), (4.29), and (4.32), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{G}\|_{L^2}^2 \leqslant -\frac{\varepsilon}{4}\|\partial_x^3 \widetilde{G}\|_{L_2}^2 + C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h(t))^2 \\ + (3d_0 - d_1 - 10b_1 M_1) \int_{\mathbb{T}} (\partial_x u_x, u_x))|\partial_x \widetilde{G}|^2 dx.\end{aligned}$$

Since $b_1 \neq 0$, we can set $M_1 = \frac{3d_0 - d_1}{10b_1}$. Then, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{G}\|_{L^2}^2 \leqslant C(\|u_x\|_{C([0,T_\varepsilon];H^7)})E(h(t))^2, \tag{4.33}$$

which is the desired result.

It is now easy to obtain the estimate for $\|h\|_{L^2}$ and $\|\partial_x h\|_{L^2}$ permitting loss of derivatives of order one. Indeed, after a lengthy calculations applying (4.10), we show that there exists a constant $C = C(\|u_x\|_{C([0,T_\varepsilon];H^7)}) > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\left\{\|h(t)\|_{L^2}^2 + \|\partial_x h(t)\|_{L^2}^2\right\} \leqslant C\, E(h(t))^2 \tag{4.34}$$

for all $t \in [0, T_\varepsilon]$. Combining (4.33) and (4.34), we derive the desired estimate (4.14). This completes the proof. $\square$

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