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HIGHER HOMOTOPY COMMUTATIVITY OF H-SPACES AND THE PERMUTO-ASSOCIAHEDRA

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Dedicated to the memory of Professor Masahiro Sugawara

Abstract. In this paper, we give a combinatorial definition of a higher homotopy commutativity of the multiplication for an A_n-space. To give the definition, we use polyhedra called the permuto-associahedra which are constructed by Kapranov. We also show that if a connected A_p-space has the finitely generated mod p cohomology for a prime p and the multiplication of it is homotopy commutative of the p-th order, then it has the mod p homotopy type of a finite product of Eilenberg-Mac Lane spaces $K(\mathbb{Z},1)$s, $K(\mathbb{Z},2)$s and $K(\mathbb{Z}/p^i,1)$s for $i \geq 1$.

1. Introduction

The notion of H-spaces was introduced to study Lie groups from a homotopy theoretic point of view. In recent decades, several theorems have been proved about the finite H-spaces (cf. [7] and [15]), which suggest that the finite H-spaces have many similar properties to those of the Lie groups.

Since being an H-space is a homotopy theoretic property, a space with the homotopy type of an H-space is also an H-space. The typical example of an H-space is a space X of the homotopy type of a loop space ΩY for some space Y. Sugawara [24] gave a criterion for a space to be of the homotopy type of a loop space. His criterion is a higher homotopy associativity of the multiplication. Later Stasheff [22] expanded the definition of Sugawara and reached the concept of the A_n-space. An A_n-space is by definition an H-space such that the multiplication is higher homotopy associative of the n-th order. The polyhedra used in his combinatorial definition are called the associahedra.

In 1960, Sugawara [25] also considered a higher homotopy commutativity of the multiplication of an associative H-space. Later Williams [26] considered another type of higher homotopy commutativity which is weaker than the one of Sugawara. In his combinatorial definition, Williams used polyhedra called the permutohedra which are originally introduced by Milgram [18] to construct approximations to the iterated loop spaces.

In the definitions by Sugawara and Williams, the multiplications of the spaces are assumed to be strictly associative. In this paper, we prove that we can define the

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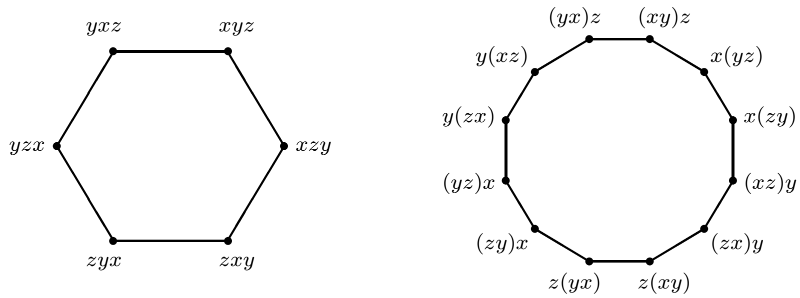

Figure 1. Higher homotopy commutativity of the third order

higher homotopy commutativity of the n-th order only assuming the multiplication is homotopy associative of the n-th order. For example, the higher homotopy commutativity of the second order is just the homotopy commutativity, so it does not need to assume any homotopy associativity on the multiplication. The higher homotopy commutativity of Williams of the third order is illustrated by the left hexagon in Figure 1. Thus, the definition is generalized for homotopy associative H-spaces by using the right dodecagon in Figure 1.

Hemmi [5] considered a generalization of the higher homotopy commutativity of Williams in the case of A_n-spaces, and introduced the quasi C_n-space. However, his definition is not a combinatorial one. Moreover, the definition of the quasi C_n-space uses the projective spaces of the multiplication of the A_n-space (see §3). Since it is not known if the projective spaces are compatible with fibrations, the quasi C_n-space is not easy to handle. For example, the authors do not know if the covering spaces inherit the property of being a quasi C_n-space.

The reason why Hemmi gave such an artificial definition is that the polyhedra used in a proper combinatorial definition become very complicated since they should be given by combining the permutohedra and the associahedra.

In 1993, such polyhedra called the permuto-associahedra were constructed by Kapranov [8]. Due to his construction, a combinatorial definition of the higher homotopy commutativity has been possible now. In the present paper, we give the combinatorial definition. An A_n-space with a multiplication of this sort is called an AC_n-space (see Definition 3.1). From the definition, X is an AC_2-space if and only if X is a homotopy commutative H-space (see Example 3.2 (1)). Moreover, our definition coincides with the one of Williams if the multiplication of the given space is strictly associative (see Corollary 3.6).

According to Hemmi [5, Prop. 2.3], a homotopy commutative H-space is a quasi C_2-space, and if the multiplication is homotopy associative, then the converse also holds (see also [23, Thm. 13.6]). In the case of AC_n-spaces, we have the following result:

Theorem A. (1) *If X is an AC_n-space, then X is a quasi C_n-space.*

(2) *If X is an A_{n+1}-space having a quasi C_n-space structure, then X is an AC_n-space.*

Since AC_n-spaces are quasi C_n-spaces, the theorems for quasi C_n-spaces are also valid for AC_n-spaces. In particular, the mod p torus theorems proved by Hemmi [5] and Kawamoto [11] are also true for AC_p-spaces. Besides, since the universal covering of an AC_n-space is also an AC_n-space (see Lemma 3.9), we have the following stronger version:

Theorem B. *Let p be a prime. If X is a connected AC_p-space such that the* mod p *cohomology $H^*(X;\mathbb{Z}/p)$ is finitely generated as an algebra, then X is* mod p *homotopy equivalent to a finite product of Eilenberg-Mac Lane spaces $K(\mathbb{Z},1)$s, $K(\mathbb{Z},2)$s and $K(\mathbb{Z}/p^i,1)$s for $i \geq 1$.*

In the above theorem, the condition of AC_p-space cannot be relaxed to AC_{p-1}-space. In fact, the odd dimensional sphere $(S^{2n-1})^\wedge_p$ completed at p is an AC_{p-1}-space for any $n \geq 1$ (see Proposition 3.8).

In the case of finite AC_p-spaces, we have the following corollary:

Corollary 1.1. *Let p be a prime. If X is a connected finite AC_p-space, then X is* mod p *homotopy equivalent to a torus.*

The above results are considered as mod p versions of the torus theorems by Hubbuck [6], Lin [13], Slack [21], Lin-Williams [16] and Broto-Crespo [3]. For the details of the mod p torus theorems, see Aguadé-Smith [1], Hemmi [5], Kawamoto [9], [10], [11], Kawamoto-Lin [12], Lin [14] and McGibbon [17]. In particular, since the loop space of an H-space is an AC_n-space for any $n \geq 1$ (see Example 3.2 (3)), we have the following result:

Theorem 1.2 ([9, Thm. A])**.** *Let p be a prime. If X is a simply connected* mod p *H-space such that the* mod p *cohomology $H^*(\Omega X;\mathbb{Z}/p)$ is finitely generated as an algebra, then ΩX is* mod p *homotopy equivalent to a finite product of Eilenberg-Mac Lane spaces $K(\mathbb{Z},1)$s, $K(\mathbb{Z},2)$s and $K(\mathbb{Z}/p^i,1)$s for $i \geq 1$.*

For the rest of this paper, all spaces are assumed to be completed at a prime p in the sense of Bousfield-Kan [2]. An H-space which is completed at p is called a mod p H-space, and it is called finite if its mod p cohomology is finite dimensional.

This paper is organized as follows: In §2, we first recall the permuto-associahedra constructed by Kapranov [8]. Then we show that the permutohedra are decomposed by using the associahedra and the permuto-associahedra (see Proposition 2.5). In §3, we give the combinatorial definition of the AC_n-form on an A_n-space by using the permuto-associahedra. Then we prove Theorem A by using the decompositions of the permutohedra in §2. By combining Theorem A with a result of Kawamoto [11] on quasi C_p-spaces with finitely generated mod p cohomology, we give the proof of Theorem B.

2. Permuto-associahedra

Stasheff [22] constructed a collection of special complexes $\{K_n\}_{n\geq 2}$ such that K_n is homeomorphic to the $(n-2)$-dimensional ball for $n \geq 2$. He used the collection $\{K_n\}_{n\geq 2}$ to introduce the higher homotopy associativity of H-spaces (see §3). The complex K_n is called the $(n-2)$-dimensional associahedron for $n \geq 2$. Let $L_n = \partial K_n$. Then by [22, p. 278],

$$L_n = \bigcup_{r,s,k} K_k(r,s)$$

for $r, s \geq 2$ with $r+s = n+1$ and $1 \leq k \leq r$. The facet (codimension one face) $K_k(r,s)$ is homeomorphic to $K_r \times K_s$ by the face operator $\partial_k(r,s) : K_r \times K_s \to K_k(r,s)$ satisfying some relations (see [22, p. 278, 3(a),(b)]). Furthermore, there is

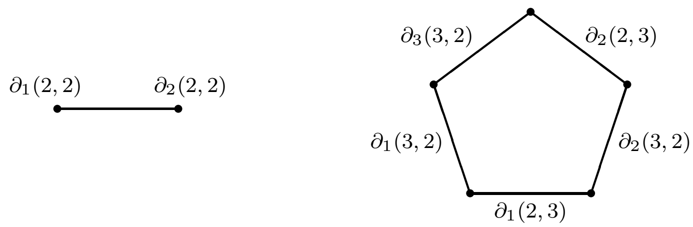

FIGURE 2. The associahedra K_3 and K_4

a collection of degeneracy operators $\{\theta_j : K_n \to K_{n-1}\}_{1\le j\le n}$ satisfying suitable conditions (see [22, I, Prop. 3]). Here Figure 2 illustrates the associahedra K_3 and K_4.

Later Kapranov [8] constructed another collection of special complexes $\{\Gamma_n\}_{n\ge1}$. By [8, Thm. 2.5], Γ_n is homeomorphic to the $(n-1)$-dimensional ball for $n \ge 1$. The complex Γ_n is closely related to the associahedra $\{K_n\}_{n\ge2}$, and is called the $(n-1)$-dimensional permuto-associahedron for $n \ge 1$ (Kapranov [8] denoted the complex by KP_n for $n \ge 1$). It is remarkable that Reiner-Ziegler [20, Thm. 2] reconstructed Γ_n as the convex hull of a finite set of points in $\mathbf{R}^n$ for $n \ge 1$ (see also Ziegler [29, Example 9.14]).

Let $\mathbf{n} = (1,\dots,n)$. For $l \ge 1$, we denote a subsequence of $\mathbf{n}$ of length l by $\alpha = (a_1,\dots,a_l)$. Let $\alpha : \mathbf{l} \to \mathbf{n}$ denote the composite $i_\alpha j_\alpha$, where $i_\alpha : \alpha \to \mathbf{n}$ is the inclusion and $j_\alpha : \mathbf{l} \to \alpha$ is the map defined by $j_\alpha(i) = a_i$ for $1 \le i \le l$. Let $t_1,\dots,t_m \ge 1$ with $t_1 + \cdots + t_m = n$. A partition of $\mathbf{n}$ of type $(t_1,\dots,t_m)$ is an ordered sequence $(\alpha_1,\dots,\alpha_m)$ consisting of disjoint subsequences α_i of length t_i for $1 \le i \le m$ with $i_{\alpha_1}(\alpha_1) \cup \cdots \cup i_{\alpha_m}(\alpha_m) = \mathbf{n}$.

From the construction of Γ_n, there is a natural way to describe all the faces of it. By [29, Def. 9.13], a facet (codimension one face) of Γ_n is represented by a partition $(\alpha_1,\dots,\alpha_m)$ of $\mathbf{n}$ with $m \ge 2$, and a codimension two face is represented by inserting a pair of parentheses in a partition $(\alpha_1,\dots,\alpha_m)$ as

$$(\alpha_1,\dots,\alpha_{i-1},(\alpha_i,\dots,\alpha_j),\alpha_{j+1},\dots,\alpha_m)$$

with $1 \le i < j \le m$. In general, a codimension $s+1$ face of Γ_n is represented by inserting s pairs of parentheses in a meaningful way to a partition $(\alpha_1,\dots,\alpha_m)$ of $\mathbf{n}$ such that any pair of parentheses includes at least two elements each of which is α_i or a parenthesized sequence. In this manner, vertices of Γ_n are represented by all meaningful complete ways of inserting parentheses to partitions of $\mathbf{n}$ of type $(1,\dots,1)$.

Now the facet of Γ_n corresponding to a partition $(\alpha_1,\dots,\alpha_m)$ is denoted by $\Gamma(\alpha_1,\dots,\alpha_m)$. Let $\Lambda_n = \partial\Gamma_n$. Then we have that

$$\Lambda_n = \bigcup_{(\alpha_1,\dots,\alpha_m)} \Gamma(\alpha_1,\dots,\alpha_m), \tag{2.1}$$

where the union covers all partitions $(\alpha_1,\dots,\alpha_m)$ of $\mathbf{n}$ for $m \ge 2$.

The permuto-associahedra Γ_2 and Γ_3 are illustrated by Figure 3 (see [29, p. 314] for the 3-dimensional permuto-associahedron Γ_4).

Here we briefly explain how to label the permuto-associahedron Γ_3 in Figure 3. Recall that the permuto-associahedra are used to describe the higher homotopy

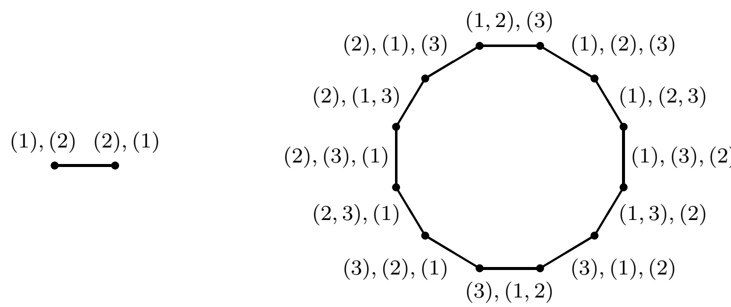

FIGURE 3. The permuto-associahedra Γ_2 and Γ_3

commutativity of A_n-spaces. When the multiplication of the space is strictly associative, the permutohedra are enough to describe it. Then we expect that the permuto-associahedra Γ_n are given by modifying the permutohedra P_n. In fact, Γ_n is given by cutting off all the faces of P_n in an appropriate way, so that each of the facets of Γ_n corresponds to one of the faces of P_n. Let us explain the case of $n = 3$. In Figure 1, the left hexagon is P_3 and the right dodecagon is Γ_3. In those pictures, we need to take x, y and z as $1, 2$ and 3, respectively. The uppermost edge is a commuting homotopy between xy and yx, and thus it is relabeled by $((1,2),(3))$. The vertex labeled by xyz in the left hexagon is relabeled by the partition $((1),(2),(3))$ in the right dodecagon. Since each of the faces of P_3 gives a facet of Γ_3, to make Γ_3, the vertex xyz in P_3 is replaced by an edge which is K_3 representing the associating homotopy between $(xy)z$ and $x(yz)$.

Now we construct the face operators for Γ_n. If $(\alpha_1, \dots, \alpha_m)$ is a partition of $\mathbf{n}$ of type $(t_1, \dots, t_m)$, then the facet $\Gamma(\alpha_1, \dots, \alpha_m)$ is homeomorphic to $K_m \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m}$ by the face operator $\epsilon^{(\alpha_1,\dots,\alpha_m)} : K_m \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m} \to \Gamma(\alpha_1, \dots, \alpha_m)$ (see [8, p. 139]).

First we give a rough idea of the construction. Again, let us explain the case of $n = 3$. In the right dodecagon of Figure 3, the uppermost edge labeled with $((1,2),(3))$ is homeomorphic to $K_2 \times \Gamma_2 \times \Gamma_1$ by the face operator $\epsilon^{((1,2),(3))}$. On the other hand, the edge labeled with $((1),(2),(3))$ is homeomorphic to $K_3 \times \Gamma_1 \times \Gamma_1 \times \Gamma_1$ by $\epsilon^{((1),(2),(3))}$. The intersection of these two edges is a vertex, the images of $(*, \epsilon^{((1),(2))}(*,*,*), *)$ by $\epsilon^{((1,2),(3))}$ and $(\partial_1(2,2)(*,*), *, *, *)$ by $\epsilon^{((1),(2),(3))}$, where $\partial_1(2,2) : K_2 \times K_2 \to K_3$ denotes a face operator of the associahedron K_3.

The next right vertex is the intersection of the two edges labeled with $((1),(2),(3))$ and $((1),(2,3))$, the images of $(\partial_2(2,2)(*,*), *, *, *)$ in $K_3 \times \Gamma_1 \times \Gamma_1 \times \Gamma_1$ by $\epsilon^{((1),(2),(3))}$ and $(*, *, \epsilon^{((1),(2))}(*,*,*))$ in $K_2 \times \Gamma_1 \times \Gamma_2$ by $\epsilon^{((1),(2,3))}$. Another next vertex, the intersection of the two edges $((1),(2,3))$ and $((1),(3),(2))$, is the images of $(*, *, \epsilon^{((2),(1))}(*,*,*))$ in $K_2 \times \Gamma_1 \times \Gamma_2$ by $\epsilon^{((1),(2,3))}$ and $(\partial_2(2,2)(*,*), *, *, *)$ in $K_3 \times \Gamma_1 \times \Gamma_1 \times \Gamma_1$ by $\epsilon^{((1),(3),(2))}$.

In Γ_4, the facet labeled with $((1,2,4),(3))$ is a dodecagon homeomorphic to $K_2 \times \Gamma_3 \times \Gamma_1$, while the facet labeled with $((1,4),(2),(3))$ is a square homeomorphic to $K_3 \times \Gamma_2 \times \Gamma_1 \times \Gamma_1$. The intersection of these two facets is an edge which is also the images of $(*, \epsilon^{((1,3),(2))}(*, t, *), *)$ by $\epsilon^{((1,2,4),(3))}$ and $(\partial_2(2,2)(*,*), t, *)$ by $\epsilon^{((1,4),(2),(3))}$ for $t \in \Gamma_2$.

In general, we have the following:

Proposition 2.1. *Let* $(\alpha_1, \dots, \alpha_m)$ *be a partition of* $\mathbf{n}$ *of type* $(t_1, \dots, t_m)$ *with* $m \geq 2$. *Then we have the following relations:*

(1) *If* $r, s \geq 2$ *with* $r + s = m + 1$ *and* $1 \leq k \leq r$, *then*

$$\begin{aligned} &\epsilon^{(\alpha_1,\dots,\alpha_m)}(\partial_k(r,s)(\rho,\sigma), \tau_1, \dots, \tau_m) \\ &= \epsilon^{(\beta_1,\dots,\beta_r)}(\rho, \tau_1, \dots, \tau_{k-1}, \epsilon^{(\gamma_1,\dots,\gamma_s)}(\sigma, \tau_k, \dots, \tau_{k+s-1}), \tau_{k+s}, \dots, \tau_m), \end{aligned}$$

where $(\beta_1, \dots, \beta_r)$ *is the partition of* $\mathbf{n}$ *of type* $(t_1, \dots, t_{k-1}, t_k + \dots + t_{k+s-1}, t_{k+s}, \dots, t_m)$ *defined by*

$$\beta_i(t) = \begin{cases} \alpha_i(t) & \text{for } 1 \leq i \leq k-1,\ 1 \leq t \leq t_i, \\ \alpha_{i+s-1}(t) & \text{for } k+1 \leq i \leq r,\ 1 \leq t \leq t_{i+s-1} \end{cases}$$

and

$$\beta_k = \alpha_k \cup \dots \cup \alpha_{k+s-1},$$

and $(\gamma_1, \dots, \gamma_s)$ *is the partition of* $(1, \dots, t_k + \dots + t_{k+s-1})$ *of type* $(t_k, \dots, t_{k+s-1})$ *given by* $\beta_k\gamma_i(t) = \alpha_{i+k-1}(t)$ *for* $1 \leq i \leq s$ *and* $1 \leq t \leq t_{i+k-1}$.

(2) *If* $(\zeta_1, \dots, \zeta_l)$ *is a partition of* $(1, \dots, t_k)$ *of type* $(u_1, \dots, u_l)$ *with* $l \geq 2$, *then*

$$\begin{aligned} &\epsilon^{(\alpha_1,\dots,\alpha_m)}(\rho, \tau_1, \dots, \tau_{k-1}, \epsilon^{(\zeta_1,\dots,\zeta_l)}(\sigma, \omega_1, \dots, \omega_l), \tau_{k+1}, \dots, \tau_m) \\ &= \epsilon^{(\eta_1,\dots,\eta_q)}(\partial_k(m,l)(\rho,\sigma), \tau_1, \dots, \tau_{k-1}, \omega_1, \dots, \omega_l, \tau_{k+1}, \dots, \tau_m), \end{aligned}$$

where $q = m+l-1$ *and* $(\eta_1, \dots, \eta_q)$ *is the partition of* $\mathbf{n}$ *of type* $(t_1, \dots, t_{k-1}, u_1, \dots, u_l, t_{k+1}, \dots, t_m)$ *defined by*

$$\eta_i(t) = \begin{cases} \alpha_i(t) & \text{for } 1 \leq i \leq k-1,\ 1 \leq t \leq t_i, \\ \alpha_k\zeta_{i-k+1}(t) & \text{for } k \leq i \leq k+l-1,\ 1 \leq t \leq u_{i-k+1}, \\ \alpha_{i-l+1}(t) & \text{for } k+l \leq i \leq q,\ 1 \leq t \leq t_{i-l+1}. \end{cases}$$

Remark 2.2. In Proposition 2.1, the statements (1) and (2) are equivalent. In fact, the partitions $(\alpha_1, \dots, \alpha_m)$, $(\beta_1, \dots, \beta_r)$ and $(\gamma_1, \dots, \gamma_s)$ in (1) correspond to the partitions $(\eta_1, \dots, \eta_q)$, $(\alpha_1, \dots, \alpha_m)$ and $(\zeta_1, \dots, \zeta_l)$ in (2), respectively.

Next we construct the degeneracy operators $\delta_j : \Gamma_n \to \Gamma_{n-1}$ for $1 \leq j \leq n$. Let e be a face of Γ_n represented by an insertion of parentheses of a partition $(\alpha_1, \dots, \alpha_m)$ of $\mathbf{n}$. To get the representation of $\delta_j(e)$, we remove j in the partition $(\alpha_1, \dots, \alpha_m)$ and replace k by $k-1$ if $k > j$. Then we modify naturally to get a parenthesized sequence in a meaningful way.

For example, if e is the edge represented by (α_1, α_2) in Γ_3 with $\alpha_1 = (1, 2)$ and $\alpha_2 = (3)$, then $\delta_2(e)$ is the vertex of Γ_2 represented by (β_1, β_2) with $\beta_1 = (1)$ and $\beta_2 = (2)$, and $\delta_3(e)$ is the edge represented by $(1, 2)$. As another example, let v be the vertex of Γ_4 represented by $(((1), (2)), ((3), (4)))$. Then $\delta_3(v)$ is the vertex of Γ_3 represented by $(((1), (2)), (3))$.

In general, we have the following result by using a similar argument to the proof of [22, I, Prop. 3] (see also [18, Lemma 4.5]):

Proposition 2.3. *Let* $n \geq 1$. *Then there is a collection of degeneracy operators* $\{\delta_j : \Gamma_n \to \Gamma_{n-1}\}_{1 \leq j \leq n}$ *satisfying the following conditions:*

(1) *Assume that* $(\alpha_1, \dots, \alpha_m)$ *is a partition of* $\mathbf{n}$ *of type* $(t_1, \dots, t_m)$ *with* $m \geq 2$. *If* $1 \leq j \leq n$, *then we can choose* $1 \leq k \leq m$ *and* $1 \leq t \leq t_k$ *with* $\alpha_k(t) = j$.

(i) *If* $t_k \geq 2$, *then*

$$\delta_j \epsilon^{(\alpha_1,\dots,\alpha_m)}(\sigma,\tau_1,\dots,\tau_m) = \epsilon^{(\tilde{\alpha}_1,\dots,\tilde{\alpha}_m)}(\sigma,\tau_1,\dots,\tau_{k-1},\delta_t(\tau_k),\tau_{k+1},\dots,\tau_m),$$

where $(\tilde{\alpha}_1,\dots,\tilde{\alpha}_m)$ *is the partition of* $(1,\dots,n-1)$ *of type* $(t_1,\dots,t_{k-1},t_k-1,t_{k+1},\dots,t_m)$ *given by*

$$\tilde{\alpha}_k(s) = \begin{cases} \alpha_k(s) & \text{if } \alpha_k(s) < j, \\ \alpha_k(s+1)-1 & \text{if } \alpha_k(s) \geq j \end{cases} \tag{2.2}$$

and for $l \neq k$,

$$\tilde{\alpha}_l(s) = \begin{cases} \alpha_l(s) & \text{if } \alpha_l(s) < j, \\ \alpha_l(s)-1 & \text{if } \alpha_l(s) > j. \end{cases} \tag{2.3}$$

(ii) *If* $m \geq 3$ *and* $t_k = 1$, *then*

$$\begin{aligned}&\delta_j \epsilon^{(\alpha_1,\dots,\alpha_m)}(\sigma,\tau_1,\dots,\tau_m)\\ &= \epsilon^{(\tilde{\alpha}_1,\dots,\tilde{\alpha}_{k-1},\tilde{\alpha}_{k+1},\dots,\tilde{\alpha}_m)}(\theta_k(\sigma),\tau_1,\dots,\tau_{k-1},\tau_{k+1},\dots,\tau_m),\end{aligned}$$

where $\theta_k : K_m \to K_{m-1}$ *denotes the degeneracy operator of the associahedron* K_m *in* [22, I, Prop. 3], *and* $(\tilde{\alpha}_1,\dots,\tilde{\alpha}_{k-1},\tilde{\alpha}_{k+1},\dots,\tilde{\alpha}_m)$ *is the partition of* $(1,\dots,n-1)$ *of type* $(t_1,\dots,t_{k-1},t_{k+1},\dots,t_m)$ *given by* (2.3).

(iii) *If* $m = 2$ *and* $t_k = 1$, *then*

$$\delta_j \epsilon^{(\alpha_1,\alpha_2)}(*,\tau_1,\tau_2) = \begin{cases} \tau_2 & \text{for } k = 1, \\ \tau_1 & \text{for } k = 2. \end{cases}$$

(2) *If* $i \leq j$, *then* $\delta_j\delta_i = \delta_i\delta_{j+1}$.

Remark 2.4. In general, $\delta_j : \Gamma_n \to \Gamma_{n-1}$ maps the two facets $((1,\dots,j-1, j+1,\dots,n),(j))$ and $((j),(1,\dots,j-1,j+1,\dots,n))$ homeomorphically onto Γ_{n-1}. On the other hand, each of the other facets in Γ_n goes to the corresponding facet in Γ_{n-1}. The reason why the first two facets map homeomorphically onto Γ_{n-1} is that those cases correspond to (iii) in Proposition 2.3 (1).

For example, $\delta_1 : \Gamma_3 \to \Gamma_2$ maps the two edges $((2,3),(1))$ and $((1),(2,3))$ of Γ_3 homeomorphically onto Γ_2, and each of other ten edges of Γ_3 corresponds to one of the vertices $((1),(2))$ and $((2),(1))$ of Γ_2.

Milgram [18] introduced the permutohedra $\{P_n\}_{n\geq 1}$ to construct approximations to the iterated loop spaces $\{\Omega^n\Sigma^n X\}_{n\geq 1}$. A few years later, Williams [26] used these complexes to define a higher homotopy commutativity of associative H-spaces.

Let $\mathbf{n} = (1,\dots,n)$. Then we can regard $\mathbf{n}$ as a point of $\mathbf{R}^n$. The symmetric group Σ_n on n letters acts on $\mathbf{R}^n$ by the permutation of the coordinates. According to Milgram [18, Def. 4.1], the permutohedron P_n is defined as the convex hull of the orbit of $\mathbf{n}$ under the action, and is homeomorphic to the $(n-1)$-dimensional ball for $n \geq 1$ (see also [26, Def. 2]). Here we illustrate the permutohedra P_2 and P_3 by Figure 4. Let $T_n = \partial P_n$. Then by [26, Thm. 3],

$$T_n = \bigcup_{(\kappa,\nu)} P(\kappa,\nu),$$

where the union covers all partitions (κ,ν) of $\mathbf{n}$ of type (u,v) for $u,v \geq 1$ with $u+v=n$. By [26, Thm. 3], the facet $P(\kappa,\nu)$ is homeomorphic to $P_u \times P_v$ by the face operator $\epsilon^{(\kappa,\nu)} : P_u \times P_v \to P(\kappa,\nu)$ (see also [18, Lemma 4.2]).

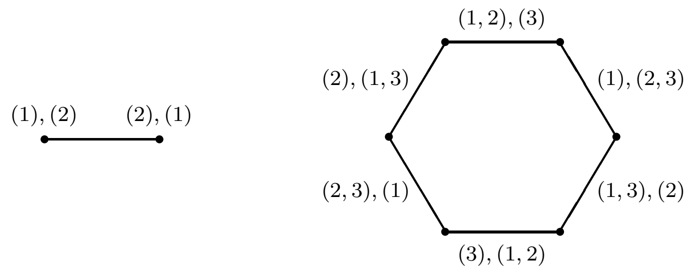

FIGURE 4. The permutohedra P_2 and P_3

Let $n \geq 1$. Assume that $(\alpha_1, \dots, \alpha_m)$ is a partition of $\mathbf{n}$ of type $(t_1, \dots, t_m)$ for $m \geq 1$ and $t_1, \dots, t_m \geq 1$ with $t_1 + \dots + t_m = n$. Let $A(\alpha_1, \dots, \alpha_m)$ be the complex defined by

$$A(\alpha_1, \dots, \alpha_m) = I \times \Gamma(\alpha_1, \dots, \alpha_m)$$

for $m \geq 2$, where I is the unit interval and $\Gamma(\alpha_1, \dots, \alpha_m)$ denotes the facet of Γ_n corresponding to the partition $(\alpha_1, \dots, \alpha_m)$ (see (2.1)). For $m = 1$, the partition $\alpha = \mathbf{n}$, and we put $A(\mathbf{n}) = \Gamma_n$.

In the proof of Proposition 3.4, we need the following result:

Proposition 2.5. *Let $n \geq 1$. Then we have the following:*

(1) *The $(n-1)$-dimensional permutohedron P_n is decomposed by*

$$P_n = \bigcup_{(\alpha_1, \dots, \alpha_m)} A(\alpha_1, \dots, \alpha_m),$$

where the union covers all partitions $(\alpha_1, \dots, \alpha_m)$ of $\mathbf{n}$ with $m \geq 1$.

(2) *If $(\alpha_1, \dots, \alpha_m)$ is a partition of $\mathbf{n}$ of type $(t_1, \dots, t_m)$, then $A(\alpha_1, \dots, \alpha_m)$ is homeomorphic to $K_{m+1} \times \Gamma_{t_1} \times \dots \times \Gamma_{t_m}$ by the operator $\iota^{(\alpha_1, \dots, \alpha_m)} : K_{m+1} \times \Gamma_{t_1} \times \dots \times \Gamma_{t_m} \to A(\alpha_1, \dots, \alpha_m)$.*

By using a similar way to the proof of [22, I, Prop. 25], we can show the following lemma:

Lemma 2.6. *There is a collection of homeomorphisms $\{\zeta_m : I \times K_m \to K_{m+1}\}_{m \geq 2}$ satisfying the following conditions:*

$$\zeta_m(0, \sigma) = \partial_1(2, m)(*, \sigma). \tag{2.4}$$

$$\zeta_m(t, \partial_k(r, s+1)(\rho, \sigma)) = \partial_k(r+1, s+1)(\zeta_r(t, \rho), \sigma) \tag{2.5}$$

for $r \geq 2$, $s \geq 1$ with $r + s = m$ and $1 \leq k \leq r$.

$$\theta_j \zeta_m(t, \sigma) = \zeta_{m-1}(t, \theta_j(\sigma)) \tag{2.6}$$

for $1 \leq j \leq m$.

Proof of Proposition 2.5. We prove by induction on n. Since $P_1 = K_2 = \Gamma_1 = *$, it is clear for $n = 1$. Now we put

$$U_n = \Gamma_n \cup_{\{0\} \times \Lambda_n} I \times \Lambda_n, \tag{2.7}$$

where $\{0\} \times \Lambda_n$ is identified with $\Lambda_n \subset \Gamma_n$. Then it is clear that U_n is homeomorphic to the $(n-1)$-dimensional ball. Now $A(\mathbf{n}) = \Gamma_n \subset U_n$. Let $\iota^{(\mathbf{n})} : K_2 \times \Gamma_n \to A(\mathbf{n})$ denote the operator given by $\iota^{(\mathbf{n})}(*, \tau) = \tau$. If $m \geq 2$, then by Lemma

2.6, we can identify the associahedron K_{m+1} with $I \times K_m$ by the homeomorphism $\zeta_m : I \times K_m \to K_{m+1}$. Assume that $(\alpha_1, \dots, \alpha_m)$ is a partition of $\mathbf{n}$ of type $(t_1, \dots, t_m)$ with $m \geq 2$. Then $A(\alpha_1, \dots, \alpha_m) = I \times \Gamma(\alpha_1, \dots, \alpha_m) \subset U_n$. Let $\iota^{(\alpha_1,\dots,\alpha_m)} : K_{m+1} \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m} \to A(\alpha_1, \dots, \alpha_m)$ denote the operator given by

$$\iota^{(\alpha_1,\dots,\alpha_m)}(\zeta_m(t, \sigma), \tau_1, \dots, \tau_m) = (t, \epsilon^{(\alpha_1,\dots,\alpha_m)}(\sigma, \tau_1, \dots, \tau_m)).$$

By (2.1) and (2.7), we see that

$$U_n = \bigcup_{(\alpha_1,\dots,\alpha_m)} A(\alpha_1, \dots, \alpha_m),$$

where the union covers all partitions $(\alpha_1, \dots, \alpha_m)$ of $\mathbf{n}$ with $m \geq 1$. If we show that U_n is the $(n-1)$-dimensional permutohedron, then we have the required conclusion.

Let $V_n = \partial U_n$. Since

$$\zeta_m(\{1\} \times K_m) = \bigcup_{r,s} K_{r+1}(r+1, s+1),$$

we have that

$$V_n = \bigcup_{(\alpha_1,\dots,\alpha_m)} \iota^{(\alpha_1,\dots,\alpha_m)} \left(\left(\bigcup_{r,s} K_{r+1}(r+1, s+1) \right) \times \Gamma_{t_1} \times \cdots \times \Gamma_{t_m} \right), \tag{2.8}$$

where $(\alpha_1, \dots, \alpha_m)$ is a partition of $\mathbf{n}$ of type $(t_1, \dots, t_m)$ with $m \geq 2$, and $r, s \geq 1$ with $r + s = m$.

To prove that V_n is homeomorphic to $T_n = \partial P_n$, we need to define a collection of face operators on V_n satisfying the conditions of [26, Thm. 3]. Assume that (κ, ν) is a partition of $\mathbf{n}$ of type (u, v). By the inductive hypothesis, there are decompositions

$$P_u = \bigcup_{(\eta_1,\dots,\eta_r)} A(\eta_1, \dots, \eta_r)$$

and

$$P_v = \bigcup_{(\lambda_1,\dots,\lambda_s)} A(\lambda_1, \dots, \lambda_s),$$

where the unions cover all partitions $(\eta_1, \dots, \eta_r)$ of $\mathbf{u}$ with $r \geq 1$ and $(\lambda_1, \dots, \lambda_s)$ of $\mathbf{v}$ with $s \geq 1$, respectively. If $(\eta_1, \dots, \eta_r)$ and $(\lambda_1, \dots, \lambda_s)$ are of types $(u_1, \dots, u_r)$ and $(v_1, \dots, v_s)$, then there are homeomorphisms $\iota^{(\eta_1,\dots,\eta_r)} : K_{r+1} \times \Gamma_{u_1} \times \cdots \times \Gamma_{u_r} \to A(\eta_1, \dots, \eta_r)$ and $\iota^{(\lambda_1,\dots,\lambda_s)} : K_{s+1} \times \Gamma_{v_1} \times \cdots \times \Gamma_{v_s} \to A(\lambda_1, \dots, \lambda_s)$. Put $m = r + s$. Let $(\alpha_1, \dots, \alpha_m)$ be the partition of $\mathbf{n}$ of type $(u_1, \dots, u_r, v_1, \dots, v_s)$ given by

$$\alpha_i(t) = \begin{cases} \kappa\eta_i(t) & \text{for } 1 \leq i \leq r,\ 1 \leq t \leq u_i, \\ \nu\lambda_{i-r}(t) & \text{for } r+1 \leq i \leq m,\ 1 \leq t \leq v_{i-r}. \end{cases}$$

If we define a face operator $\epsilon^{(\kappa,\nu)} : P_u \times P_v \to V_n$ by

$$\begin{aligned} &\epsilon^{(\kappa,\nu)}(\iota^{(\eta_1,\dots,\eta_r)}(\rho, \tau_1, \dots, \tau_r), \iota^{(\lambda_1,\dots,\lambda_s)}(\sigma, \omega_1, \dots, \omega_s)) \\ &= \iota^{(\alpha_1,\dots,\alpha_m)}(\partial_{r+1}(r+1, s+1)(\rho, \sigma), \tau_1, \dots, \tau_r, \omega_1, \dots, \omega_s), \end{aligned}$$

then by (2.8),

$$V_n = \bigcup_{(\kappa,\nu)} \epsilon^{(\kappa,\nu)}(P_u \times P_v),$$

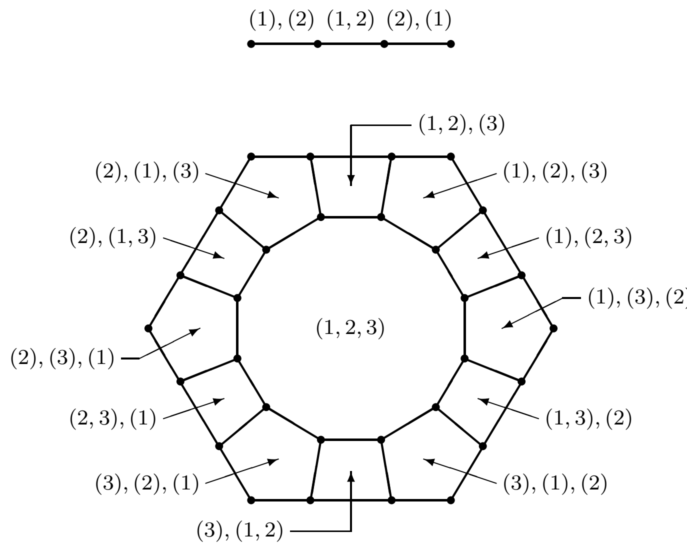

FIGURE 5. The decompositions of the permutohedra P_2 and P_3

where the union covers all partitions (κ, ν) of $\mathbf{n}$. By using the relation

$$\begin{aligned}&\partial_{r+s+1}(r+s+1,t+1)(\partial_{r+1}(r+1,s+1)\times 1_{K_{t+1}})\\ &=\partial_{r+1}(r+1,s+t+1)(1_{K_{r+1}}\times\partial_{s+1}(s+1,t+1))\end{aligned}$$

for $r, s, t \geq 1$, we can show that the collection of face operators satisfies the conditions of [26, Thm. 3], and so V_n is homeomorphic to $T_n = \partial P_n$. This implies that U_n is the $(n-1)$-dimensional permutohedron, and we have the required conclusion. For example, the decompositions of the permutohedra P_2 and P_3 are illustrated by Figure 5. This completes the proof. □

A collection of degeneracy operators $\{\xi_j : P_n \to P_{n-1}\}_{1\leq j\leq n}$ for the permutohedra is originally constructed by Milgram [18, Lemma 4.5] (see also [26, Lemma 4]). By using Proposition 2.5, we give another construction of $\{\xi_j\}_{1\leq j\leq n}$ which is useful for our arguments.

By Proposition 2.5, there is a decomposition

$$P_n = \bigcup_{(\alpha_1,\ldots,\alpha_m)} A(\alpha_1, \ldots, \alpha_m),$$

where the union covers all partitions $(\alpha_1, \ldots, \alpha_m)$ of $\mathbf{n}$ with $m \geq 1$. Let $(\alpha_1, \ldots, \alpha_m)$ be a partition of $\mathbf{n}$ of type $(t_1, \ldots, t_m)$. If $1 \leq j \leq n$, then $\alpha_k(t) = j$ for some $1 \leq k \leq m$ and $1 \leq t \leq t_k$. The map $\xi_j : P_n \to P_{n-1}$ is defined by

$$\begin{aligned}&\xi_j\iota^{(\alpha_1,\ldots,\alpha_m)}(\rho,\tau_1,\ldots,\tau_m)\\ &=\begin{cases}\iota^{(\tilde\alpha_1,\ldots,\tilde\alpha_m)}(\rho,\tau_1,\ldots,\tau_{k-1},\delta_t(\tau_k),\tau_{k+1},\ldots,\tau_m) & \text{if } t_k\geq 2,\\ \iota^{(\tilde\alpha_1,\ldots,\tilde\alpha_{k-1},\tilde\alpha_{k+1},\ldots,\tilde\alpha_m)}(\theta_k(\rho),\tau_1,\ldots,\tau_{k-1},\tau_{k+1},\ldots,\tau_m) & \text{if } t_k=1,\end{cases}\end{aligned}$$

where the partitions $(\tilde\alpha_1, \ldots, \tilde\alpha_m)$ and $(\tilde\alpha_1, \ldots, \tilde\alpha_{k-1}, \tilde\alpha_{k+1}, \ldots, \tilde\alpha_m)$ are defined by (2.2)–(2.3), and $\{\delta_t\}_{1\leq t\leq t_k}$ and $\{\theta_k\}_{1\leq k\leq m}$ denote the degeneracy operators for Γ_{t_k} and K_{m+1}, respectively. Since we can show that $\{\xi_j\}_{1\leq j\leq n}$ satisfies the conditions of [18, Lemma 4.5] (see also [26, Lemma 4]), $\{\xi_j\}_{1\leq j\leq n}$ is a collection of degeneracy operators for the permutohedra $\{P_n\}_{n\geq 1}$.

3. Proofs of Theorem A and Theorem B

Stasheff [22] introduced the notion of the higher homotopy associativity of H-spaces. He used the associahedra $\{K_i\}_{2\leq i\leq n}$ to define an A_n-form on an H-space. Let $n \geq 2$ and X be an H-space with a multiplication $\mu : X \times X \to X$ such that $\mu(x, *) = \mu(*, x) = x$ for $x \in X$. An A_n-form on X is a collection of maps $\{M_i : K_i \times X^i \to X\}_{2\leq i\leq n}$ satisfying the following conditions:

$$M_2(*, x, y) = \mu(x, y). \tag{3.1}$$

$$\begin{aligned} &M_i(\partial_k(r,s)(\rho,\sigma), x_1,\dots,x_i)\\ &= M_r(\rho, x_1,\dots,x_{k-1}, M_s(\sigma, x_k,\dots,x_{k+s-1}), x_{k+s},\dots,x_i), \end{aligned} \tag{3.2}$$

where $r, s \geq 2$ with $r + s = i + 1$ and $1 \leq k \leq r$.

$$\begin{aligned} &M_i(\tau, x_1,\dots,x_{j-1}, *, x_{j+1},\dots,x_i)\\ &= M_{i-1}(\theta_j(\tau), x_1,\dots,x_{j-1}, x_{j+1},\dots,x_i), \end{aligned} \tag{3.3}$$

where $\{\theta_j : K_i \to K_{i-1}\}_{1\leq j\leq i}$ are the degeneracy operators. For convenience, we define that an A_1-space is just a space. For $n \geq 2$, an A_n-space is an H-space X with a specified A_n-form on X. If X has a collection of maps $\{M_i : K_i \times X^i \to X\}_{i\geq 2}$ such that $\{M_i\}_{2\leq i\leq n}$ is an A_n-form on X for any $n \geq 2$, then X is called an A_∞-space. From the definition of an A_n-form, we see that an A_2-space and an A_3-space are an H-space and a homotopy associative H-space, respectively. Furthermore, an A_∞-space has the homotopy type of an associative H-space.

Now we introduce the higher homotopy commutativity of H-spaces. An AC_n-form on an A_n-space is defined by using a collection of the permuto-associahedra $\{\Gamma_i\}_{1\leq i\leq n}$.

Definition 3.1. Let X be an A_n-space with the A_n-form $\{M_i\}_{2\leq i\leq n}$. An AC_n-form on X consists of a collection of maps $\{Q_i : \Gamma_i \times X^i \to X\}_{1\leq i\leq n}$ satisfying the following conditions:

$$Q_1(*, x) = x. \tag{3.4}$$

$$\begin{aligned} &Q_i(\epsilon^{(\alpha_1,\dots,\alpha_m)}(\sigma,\tau_1,\dots,\tau_m), x_1,\dots,x_i)\\ &= M_m(\sigma, Q_{t_1}(\tau_1, x_{\alpha_1(1)},\dots,x_{\alpha_1(t_1)}),\dots,Q_{t_m}(\tau_m, x_{\alpha_m(1)},\dots,x_{\alpha_m(t_m)})), \end{aligned} \tag{3.5}$$

where $(\alpha_1,\dots,\alpha_m)$ is a partition of $\mathbf{i}$ of type $(t_1,\dots,t_m)$.

$$\begin{aligned} &Q_i(\tau, x_1,\dots,x_{j-1}, *, x_{j+1},\dots,x_i)\\ &= Q_{i-1}(\delta_j(\tau), x_1,\dots,x_{j-1}, x_{j+1},\dots,x_i), \end{aligned} \tag{3.6}$$

where $\{\delta_j : \Gamma_i \to \Gamma_{i-1}\}_{1\leq j\leq i}$ are the degeneracy operators.

An A_n-space with a specified AC_n-form is called an AC_n-space. If X has a collection of maps $\{Q_i : \Gamma_i \times X^i \to X\}_{i\geq 1}$ such that $\{Q_i\}_{1\leq i\leq n}$ is an AC_n-form on X for any $n \geq 1$, then X is called an AC_∞-space.

Example 3.2. (1) X is an AC_2-space if and only if X is a homotopy commutative H-space since $Q_2(\epsilon^{((1),(2))}(*), x, y) = xy$ and $Q_2(\epsilon^{((2),(1))}(*), x, y) = yx$ for $x, y \in X$.

(2) If X is an associative and commutative H-space, then the collection $\{Q_i : \Gamma_i \times X^i \to X\}_{i\geq 1}$ defined by $Q_i(\tau, x_1,\dots,x_i) = x_1 \dots x_i$ for $i \geq 1$ makes X an

AC_∞-space. In particular, Eilenberg-Mac Lane spaces are AC_∞-spaces by Stasheff [23, Cor. 13.10].

(3) If X is an H-space, then by Corollary 3.6 and [26, Cor. 26], ΩX is an AC_∞-space.

Now we recall the definition of a quasi C_n-form on an A_n-space introduced by Hemmi [5, Def. 2.1]. Let X be an A_n-space and $\{P_i(X)\}_{1\le i\le n}$ be the projective spaces of X. From the construction of $P_i(X)$, we have the inclusion $\iota_{i-1} : P_{i-1}(X) \to P_i(X)$ and the projection $\rho_i : P_i(X) \to P_i(X)/P_{i-1}(X) \simeq (\Sigma X)^{(i)}$, where $(\Sigma X)^{(i)}$ denotes the i-fold smash product of ΣX for $1 \le i \le n$. Let $J_i(\Sigma X)$ denote the i-th James reduced product space of ΣX and $\pi_i : J_i(\Sigma X) \to (\Sigma X)^{(i)}$ be the obvious projection for $1 \le i \le n$. A quasi C_n-form on X is a collection of maps $\{\psi_i : J_i(\Sigma X) \to P_i(X)\}_{1\le i\le n}$ satisfying the following conditions:

$$\psi_1 = 1_{\Sigma X} : \Sigma X \longrightarrow \Sigma X. \tag{3.7}$$

$$\psi_i|_{J_{i-1}(\Sigma X)} = \iota_{i-1}\psi_{i-1} \qquad \text{for } 2 \le i \le n. \tag{3.8}$$

$$\rho_i\psi_i \simeq \left(\sum_{\sigma\in\Sigma_i} \sigma\right)\pi_i \qquad \text{for } 1 \le i \le n, \tag{3.9}$$

where the action of the symmetric group Σ_i on $(\Sigma X)^{(i)}$ is given by the permutation of the coordinates, and the summation on the right hand side is defined by using the obvious co-H-structure on $(\Sigma X)^{(i)}$. An A_n-space with a specified quasi C_n-form is called a quasi C_n-space. Hemmi [5, Thm. 1.1] has shown that a simply connected finite quasi C_p-space is contractible. Furthermore, Kawamoto [11] generalized the result to the case of quasi C_p-spaces with finitely generated mod p cohomology.

Theorem 3.3 ([11, Thm. B])**.** *If X is a simply connected quasi C_p-space such that the* mod p *cohomology $H^*(X;\mathbb{Z}/p)$ is finitely generated as an algebra, then X is* mod p *homotopy equivalent to a finite product of $K(\mathbb{Z},2)$s.*

To prove Theorem A, we define a C_n-form of a map from a space to a loop space by using the permutohedra $\{P_i\}_{1\le i\le n}$.

Let X and Y be spaces and $\phi : X \to \Omega Y$ be a map. A C_n-form on ϕ is a collection of maps $\{R_i : P_i \times X^i \to \Omega Y\}_{1\le i\le n}$ satisfying the following conditions:

$$R_1(*, x) = \phi(x). \tag{3.10}$$

$$\begin{aligned} &R_i(\epsilon^{(\alpha,\beta)}(\rho,\sigma), x_1, \dots, x_i) \\ &= R_r(\rho, x_{\alpha(1)}, \dots, x_{\alpha(r)}) \cdot R_s(\sigma, x_{\beta(1)}, \dots, x_{\beta(s)}), \end{aligned} \tag{3.11}$$

where (α,β) is a partition of $\mathbf{i}$ of type (r,s).

$$\begin{aligned} &R_i(\tau, x_1, \dots, x_{j-1}, *, x_{j+1}, \dots, x_i) \\ &= R_{i-1}(\xi_j(\tau), x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i), \end{aligned} \tag{3.12}$$

where $\{\xi_j : P_i \to P_{i-1}\}_{1\le j\le i}$ are the degeneracy operators.

Now we prove the following result:

Proposition 3.4. *Let X be an A_n-space and $\phi_n : X \to \Omega P_n(X)$ denote the adjoint of the inclusion $\iota_{n-1}\dots\iota_1 : \Sigma X \to P_n(X)$. If X admits an AC_n-form, then there is a C_n-form $\{R_i : P_i \times X^i \to \Omega P_n(X)\}_{1\le i\le n}$ on ϕ_n.*

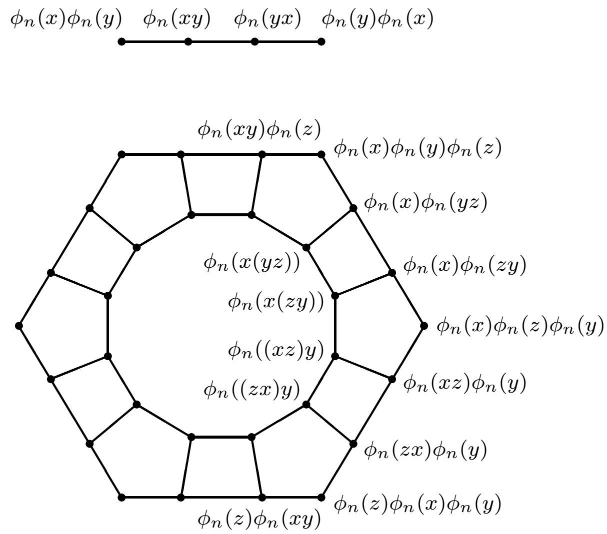

FIGURE 6. The C_n-forms on ϕ_n for $n = 2, 3$

Proof. We prove by induction on n. For $n = 1$, it is clear by (3.10). Suppose that we have a C_{n-1}-form $\{\tilde{R}_i\}_{1\leq i\leq n-1}$ on ϕ_{n-1}. If we put that $R_i = \iota_{n-1}\tilde{R}_i$ for $1 \leq i \leq n-1$, then the collection $\{R_i\}_{1\leq i\leq n-1}$ is a C_{n-1}-form on ϕ_n. According to Stasheff [23, Thm. 11.10], $\phi_n : X \to \Omega P_n(X)$ is an A_n-map, and so there is a collection of maps $\{F_i : K_{i+1} \times X^i \to \Omega P_n(X)\}_{1\leq i\leq n}$ satisfying the following conditions:

$$F_1(*, x) = \phi_n(x). \tag{3.13}$$

$$\begin{aligned} &F_i(\partial_k(r+1, s+1)(\rho, \sigma), x_1, \ldots, x_i) \\ &\quad = \begin{cases} F_r(\rho, x_1, \ldots, x_{k-1}, M_{s+1}(\sigma, x_k, \ldots, x_{k+s}), x_{k+s+1}, \ldots, x_i) & \text{if } 1 \leq k \leq r, \\ F_r(\rho, x_1, \ldots, x_r) \cdot F_s(\sigma, x_{r+1}, \ldots, x_i) & \text{if } k = r+1, \end{cases} \end{aligned}$$

where $r, s \geq 1$ with $r + s = i$ and $1 \leq k \leq r+1$.

$$\begin{aligned} &F_i(\tau, x_1, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_i) \\ &\quad = F_{i-1}(\theta_j(\tau), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i) \end{aligned}$$

for $1 \leq j \leq i$.

By Proposition 2.5, there is a decomposition

$$P_n = \bigcup_{(\alpha_1,\ldots,\alpha_m)} A(\alpha_1, \ldots, \alpha_m),$$

where the union covers all partitions $(\alpha_1, \ldots, \alpha_m)$ of $\mathbf{n}$ with $m \geq 1$. If we define a map $R_n : P_n \times X^n \to \Omega P_n(X)$ by

$$\begin{aligned} &R_n(\iota^{(\alpha_1,\ldots,\alpha_m)}(\sigma, \tau_1, \ldots, \tau_m), x_1, \ldots, x_n) \\ &\quad = F_m(\sigma, Q_{t_1}(\tau_1, x_{\alpha_1(1)}, \ldots, x_{\alpha_1(t_1)}), \ldots, Q_{t_m}(\tau_m, x_{\alpha_m(1)}, \ldots, x_{\alpha_m(t_m)})), \end{aligned}$$

then the collection $\{R_i\}_{1\leq i\leq n}$ satisfies the conditions (3.10)–(3.12). For example, the C_n-forms on ϕ_n for $n = 2, 3$ are illustrated by Figure 6. This completes the proof. □

Now we proceed to the proof of Theorem A.

Proof of Theorem A. First we prove (1) by induction on n. For $n = 1$, it is clear by (3.7). Suppose that we have a quasi C_{n-1}-form $\{\psi_i\}_{1\le i\le n-1}$ on X. By Proposition 3.4, there is a C_n-form $\{R_i\}_{1\le i\le n}$ on ϕ_n. For the same reason as in [28], we can assume without loss of generality that the image of R_i is contained in the set of loops of length i for $1 \le i \le n$. Let $\zeta_i : [0,i] \times P_i \times X^i \to P_n(X)$ be the adjoint of R_i for $1 \le i \le n$. It is shown by Williams [27] that there is a map $\tau_i : [0,i] \times P_i \to I^i$ satisfying suitable conditions (see also Milgram [18, Lemma 4.6]). Let $\kappa_i : [0,i] \times P_i \times X^i \to J_i(\Sigma X)$ be the map defined by $\kappa_i = \chi_i(\tau_i \times 1_{X^i})$, where $\chi_i : I^i \times X^i \to J_i(\Sigma X)$ denotes the obvious projection for $1 \le i \le n$. Then by [28, Thm. 1.1], we have a map $\psi_n : J_n(\Sigma X) \to P_n(X)$ which satisfies (3.8) and $\psi_n\kappa_n = \zeta_n$.

Here we explain the construction of ψ_n briefly since [28, Thm. 1.1] omitted the proof. By the inductive hypothesis, we can assume that $\psi_i\kappa_i = \zeta_i$ for $1 \le i \le n-1$. It is known that $J_n(\Sigma X) = J_{n-1}(\Sigma X) \cup_{\eta_n} I^n \times X^n$, where $\eta_n : \partial I^n \times X^n \cup I^n \times X^{[n]} \to J_{n-1}(\Sigma X)$ is the map defined by

$$\eta_n(t_1,\dots,t_n,x_1,\dots,x_n) = ((t_1,x_1),\dots,(t_{i-1},x_{i-1}),(t_{i+1},x_{i+1}),\dots,(t_n,x_n))$$

if $t_i \in \partial I$ or $x_i = *$ for $1 \le i \le n$, and $X^{[n]}$ denotes the n-fold fat wedge of X given by

$$X^{[n]} = \{(x_1,\dots,x_n) \in X^n \mid x_i = * \text{ for some } 1 \le i \le n\}.$$

Let $T_n = [0,n]\times P_n\times X^n$ and $S_n = \partial([0,n]\times P_n)\times X^n \cup [0,n]\times P_n\times X^{[n]}$. If we define a map $\lambda_n : S_n \to J_{n-1}(\Sigma X)$ by $\lambda_n = \eta_n(\tau_n \times 1_{X^n})|_{S_n}$, then $\iota_{n-1}\psi_{n-1}\lambda_n = \zeta_n|_{S_n}$, and so there is a map $\theta_n : J_{n-1}(\Sigma X)\cup_{\lambda_n} T_n \to P_n(X)$ with $\theta_n|_{J_{n-1}(\Sigma X)} = \iota_{n-1}\psi_{n-1}$ and $\theta_n|_{T_n} = \zeta_n$. Since there is a homotopy equivalence $\nu_n : J_{n-1}(\Sigma X) \cup_{\lambda_n} T_n \to J_n(\Sigma X)$ with $\nu_n|_{J_{n-1}(\Sigma X)} = \epsilon_{n-1}$ and $\nu_n|_{T_n} = \kappa_n$, we have a map $\psi_n : J_n(\Sigma X) \to P_n(X)$ such that $\psi_n|_{J_{n-1}(\Sigma X)} = \iota_{n-1}\psi_{n-1}$ and $\psi_n\kappa_n \simeq \zeta_n$ rel S_n. By replacing ζ_n with $\tilde{\zeta}_n = \psi_n\kappa_n$, we have the required conclusion.

Now we consider the condition (3.9). It is sufficient to show that

$$\rho_n\zeta_n \simeq \left(\sum_{\sigma\in\Sigma_n} \sigma\right)\pi_n\kappa_n$$

since $\zeta_n = \psi_n\kappa_n$. From the proof of Proposition 3.4, the map $R_n : P_n \times X^n \to \Omega P_n(X)$ is constructed by using the AC_n-form $\{Q_i\}_{1\le i\le n}$ on X and the A_n-form $\{F_i\}_{1\le i\le n}$ on ϕ_n. Since the image of F_i is contained in $\Omega P_{n-1}(X)$ for $1 \le i \le n-1$ from the proof of [23, Thm. 11.10], we have that

$$(\Omega\rho_n)R_n \simeq \left(\bigvee_{\sigma\in\Sigma_n} G_\sigma\right) H_n.$$

Here $G_\sigma : \Sigma^{n-1}X^{(n)} \to \Omega P_n(X)$ is represented by the map $\tilde{G}_\sigma : P_n\times X^n \to \Omega P_n(X)$ given by $\tilde{G}_\sigma(\tau,x_1,\dots,x_n) = F_n(\tau,x_{\sigma(1)},\dots,x_{\sigma(n)})$ for $\sigma \in \Sigma_n$, and

$$H_n : P_n \times X^n \to \bigvee_{\sigma\in\Sigma_n} \Sigma^{n-1}X^{(n)}$$

denotes the appropriate collapsing map. From these observations, we have the condition (3.9).

Next let us prove (2). For $n = 1$, it is clear by (3.4). By the inductive hypothesis, we assume that there is an AC_{n-1}-form $\{Q_i\}_{1\le i\le n-1}$ on X.

Now we use a result of Williams [27] on the relation between higher homotopy commutativity and extension of maps. Let $g : (\Sigma X)^{\vee n} \to P_n(X)$ be the map defined by $g|_{\Sigma X} = \iota_{n-1}\ldots\iota_1$ for each factor, where $(\Sigma X)^{\vee n}$ denotes the n-fold wedge of ΣX. Since X is a quasi C_n-space, g is extended to a map $\tilde{g} : (\Sigma X)^n \to P_n(X)$ by using ψ_n. Then by [27, Thm. 2], there exists a C_n-form $\{R_i\}_{1\le i\le n}$ on ϕ_n. Now we can assume without loss of generality that the C_{n-1}-form $\{R_i\}_{1\le i\le n-1}$ is obtained from $\{Q_i\}_{1\le i\le n-1}$ by using the way in the proof of Proposition 3.4. Since ϕ_n is an A_n-map, we have a map $\tilde{Q}_n : \Gamma_n \times X^n \to \Omega P_n(X)$ such that the $\{\tilde{Q}_i\}_{1\le i\le n}$ satisfy the conditions (3.5)–(3.6), where $\tilde{Q}_i = \phi_n Q_i$ for $1 \le i \le n-1$. Since X is an A_{n+1}-space, there is a map $\omega_n : \Omega P_n(X) \to X$ with $\omega_n\phi_n = 1_X$. Put $Q_n = \omega_n\tilde{Q}_n : \Gamma_n \times X^n \to X$. Then $\{Q_i\}_{1\le i\le n}$ is an AC_n-form on X, and so we have the required conclusion. This completes the proof of Theorem A. □

Remark 3.5. The decompositions of the permutohedra in Proposition 2.5 play important roles in the proof of Theorem A. We note that Hemmi [5] gave another decomposition of P_n by using the permutohedra $\{P_i\}_{1\le i\le n-1}$ and the simplices $\{\Delta^i\}_{1\le i\le n-1}$. He used the result to show that if X is an associative H-space, then the quasi C_n-form on X is equivalent to a C_n-form in the sense of Williams [26, Def. 5] (see [5, Thm. 2.2]). The proof of Theorem A is regarded as a generalization of the one of [5, Thm. 2.2] to the case of A_n-spaces.

From Theorem A and the result by Hemmi [5, Thm. 2.2], we have the following result:

Corollary 3.6. *Let X be an associative H-space. Then X is an AC_n-space if and only if X is a C_n-space in the sense of Williams.*

It is natural to consider the notion of maps between AC_n-spaces preserving AC_n-forms.

Let X and Y be A_n-spaces. According to Stasheff [22, II, Def. 4.1], a map $\phi : X \to Y$ is called an A_n-homomorphism if $\phi M_i^X = M_i^Y(1_{K_i} \times \phi^i)$ for $2 \le i \le n$, where $\{M_i^X\}_{2\le i\le n}$ and $\{M_i^Y\}_{2\le i\le n}$ are A_n-forms on X and Y, respectively.

Definition 3.7. Let X and Y be AC_n-spaces with the AC_n-forms $\{Q_i^X\}_{1\le i\le n}$ and $\{Q_i^Y\}_{1\le i\le n}$, respectively. An A_n-homomorphism $\phi : X \to Y$ is called an AC_n-homomorphism if $\phi Q_i^X = Q_i^Y(1_{\Gamma_i} \times \phi^i)$ for $1 \le i \le n$.

Now we consider the odd dimensional sphere $(S^{2n-1})^\wedge_p$ completed at p for $n \ge 1$. Let $\epsilon_n : (S^{2n-1})^\wedge_p \to \Omega^2(S^{2n+1})^\wedge_p$ denote the double suspension which is the double adjoint of the identity $1_{(S^{2n+1})^\wedge_p}$ on $(S^{2n+1})^\wedge_p \simeq \Sigma^2(S^{2n-1})^\wedge_p$. By Example 3.2 (3), $\Omega^2(S^{2n+1})^\wedge_p$ is an AC_∞-space. According to Stasheff [22, I, Thm. 17], $(S^{2n-1})^\wedge_p$ admits an A_{p-1}-form so that $\epsilon_n : (S^{2n-1})^\wedge_p \to \Omega^2(S^{2n+1})^\wedge_p$ is an A_{p-1}-homomorphism. By using a similar argument to the proof of [22, I, Thm. 17], we can prove the following result:

Proposition 3.8. *Let p be a prime. Then $(S^{2n-1})^\wedge_p$ admits an AC_{p-1}-form so that the double suspension $\epsilon_n : (S^{2n-1})^\wedge_p \to \Omega^2(S^{2n+1})^\wedge_p$ is an AC_{p-1}-homomorphism for $n \ge 1$.*

In the proof of Theorem B, we need the following lemma:

Lemma 3.9. *If X is a connected AC_n-space, then the universal covering $\tilde{X}$ is a simply connected AC_n-space and the covering projection map $\omega : \tilde{X} \to X$ is an AC_n-homomorphism.*

Proof. We give an outline of the proof. Let $\{M_i\}_{2\leq i\leq n}$ and $\{Q_i\}_{1\leq i\leq n}$ be the A_n-form and the AC_n-form on X, respectively. From the covering lifting property (cf. [19, Ch. 2, Lemma 1.7]), there are maps $\tilde{M}_i : K_i \times \tilde{X}^i \to \tilde{X}$ and $\tilde{Q}_i : \Gamma_i \times \tilde{X}^i \to \tilde{X}$ such that $\omega\tilde{M}_i = M_i(1_{K_i} \times \omega^i)$ for $2 \leq i \leq n$ and $\omega\tilde{Q}_i = Q_i(1_{\Gamma_i} \times \omega^i)$ for $1 \leq i \leq n$. From the uniqueness of the lifting, the collections $\{\tilde{M}_i\}_{2\leq i\leq n}$ and $\{\tilde{Q}_i\}_{1\leq i\leq n}$ satisfy the conditions (3.1)–(3.3) and (3.4)–(3.6), respectively. This completes the proof. □

Now we proceed to the proof of Theorem B.

Proof of Theorem B. Let X be a connected AC_p-space with finitely generated mod p cohomology. If $\tilde{X}$ denotes the universal covering of X, then there is an H-fibration

$$\tilde{X} \longrightarrow X \longrightarrow K(\pi_1(X), 1), \tag{3.14}$$

where $K(\pi_1(X), 1)$ has the mod p homotopy type of a finite product of $K(\mathbb{Z}, 1)$s and $K(\mathbb{Z}/p^i, 1)$s for $i \geq 1$. According to Browder [4], there is a version of the Serre spectral sequence associated to the H-fibration (3.14). As in the argument of [7, §3], we see that the mod p cohomology $H^*(\tilde{X}; \mathbb{Z}/p)$ is finitely generated as an algebra. From Theorem A, Theorem 3.3 and Lemma 3.9, $\tilde{X}$ is mod p homotopy equivalent to a finite product of $K(\mathbb{Z}, 2)$s. For dimensional reasons, the spectral sequence associated to the H-fibration (3.14) collapses. Hence we have that

$$H^*(X; \mathbb{Z}/p) \cong H^*(K(\pi_1(X), 1); \mathbb{Z}/p) \otimes H^*(\tilde{X}; \mathbb{Z}/p),$$

and there is a map $\zeta : X \to K(\pi_1(X), 1) \times \tilde{X}$ which induces an isomorphism on the mod p cohomology. Then ζ is a mod p homotopy equivalence (cf. [19, Ch. 4, Cor. 1.6]), and so we have the required conclusion. This completes the proof of Theorem B. □

References

[1] J. Aguadé and L. Smith, *On the* mod p *torus theorem of John Hubbuck*, Math. Z. **191** (1986), 325–326. MR 87e:57044

[2] A. Bousfield and D. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. **304**, Springer-Verlag, 1972. MR 51:1825

[3] C. Broto and J. A. Crespo, *H-spaces with noetherian mod two cohomology algebra*, Topology **38** (1999), 353–386. MR 99i:55013

[4] W. Browder, *The cohomology of covering spaces of H-spaces*, Bull. Amer. Math. Soc. **65** (1959), 140–141. MR 22:1891

[5] Y. Hemmi, *Higher homotopy commutativity of H-spaces and the* mod p *torus theorem*, Pacific J. Math. **149** (1991), 95–111. MR 92a:55010

[6] J. R. Hubbuck, *On homotopy commutative H-spaces*, Topology **8** (1969), 119–126. MR 38:6592

[7] R. M. Kane, *The homology of Hopf spaces*, North-Holland Math. Library **40**, North-Holland, 1988. MR 90f:55018

[8] M. M. Kapranov, *The permutoassociahedron, Mac Lane's coherence theorem and asymptotic zones for the KZ equation*, J. Pure Appl. Algebra **85** (1993), 119–142. MR 94b:52017

[9] Y. Kawamoto, *Loop spaces of H-spaces with finitely generated cohomology*, Pacific J. Math. **190** (1999), 311–328. MR 2000i:55027

[10] ———, *Homotopy classification of higher homotopy commutative loop spaces with finitely generated cohomology*, Hiroshima Math. J. **30** (2000), 317–344. MR 2001e:55011
[11] ———, *Higher homotopy commutativity of H-spaces with finitely generated cohomology*, Pacific J. Math. **204** (2002), 145–161. MR 2003d:55010
[12] Y. Kawamoto and J. P. Lin, *Homotopy commutativity of H-spaces with finitely generated cohomology*, Trans. Amer. Math. Soc. **353** (2001), 4481–4496. MR 2002g:55017
[13] J. P. Lin, *A cohomological proof of the torus theorem*, Math. Z. **190** (1985), 469–476. MR 87c:55009
[14] ———, *Loops of H-spaces with finitely generated cohomology rings*, Topology Appl. **60** (1994), 131–152. MR 95m:55022
[15] ———, *H-spaces with finiteness conditions*, Handbook of Algebraic Topology, edited by I.M. James, North-Holland, 1995, 1095–1141. MR 97c:55017
[16] J. P. Lin and F. D. Williams, *Homotopy-commutative H-spaces*, Proc. Amer. Math. Soc. **113** (1991), 857–865. MR 92b:55011
[17] C. A. McGibbon, *Higher forms of homotopy commutativity and finite loop spaces*, Math. Z. **201** (1989), 363–374. MR 90f:55019
[18] R. J. Milgram, *Iterated loop spaces,* Ann. of Math. **84** (1966), 386–403. MR 34:6767
[19] M. Mimura and H. Toda, *Topology of Lie groups*, I and II, Trans. Math. Monographs **91**, Amer. Math. Soc., 1991. MR 92h:55001
[20] V. Reiner and G. M. Ziegler, *Coxeter-associahedra*, Mathematika **41** (1994), 364–393. MR 95m:52023
[21] M. Slack, *A classification theorem for homotopy commutative* mod 2 *H-spaces with finitely generated cohomology rings*, Mem. Amer. Math. Soc. **92** (1991). MR 92k:55015
[22] J. D. Stasheff, *Homotopy associativity of H-spaces*, I and II, Trans. Amer. Math. Soc. **108** (1963), 275–292, 293–312. MR 28:1623
[23] ———, *H-spaces from a homotopy point of view*, Lecture Notes in Math. **161**, Springer-Verlag, 1970. MR 42:5261
[24] M. Sugawara, *A condition that a space is group-like*, Math. J. Okayama Univ. **7** (1957), 123–149. MR 20:3546
[25] ———, *On the homotopy commutativity of groups and loop spaces*, Mem. College Sci. Univ. Kyoto Ser. A **33** (1960), 257–269. MR 22:11394
[26] F. D. Williams, *Higher homotopy-commutativity*, Trans. Amer. Math. Soc. **139** (1969), 191–206. MR 39:2163
[27] ———, *Higher homotopy commutativity and extension of maps*, Proc. Amer. Math. Soc. **26** (1970), 664–670. MR 42:8491
[28] ———, *Higher Samelson products*, J. Pure Appl. Algebra **2** (1972), 249–260. MR 46:2670
[29] G. M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Math. **152**, Springer-Verlag, 1994. MR 96a:52011

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