Stable homotopy categories and stable homotopy groups of spheres

Ryo Kato

Preface

In this thesis, we study the stable homotopy category $\mathcal{S}$ of spectra. For a spectrum E, we have the *E_*-homology theory.* Bousfield defined a localization functor $L_E\colon \mathcal{S} \to \mathcal{S}$ with respect to a spectrum E, which classifies spectra by the E_*-homology theory. Furthermore, for a spectrum E, Bousfield defined a class $\langle E \rangle$, which is called the *Bousfield class of* E, such that $L_E = L_F$ if and only if $\langle E \rangle = \langle F \rangle$. Bousfield also studied the lattice structure of these classes. Ohkawa showed that these classes form a set, which implies the classes form a lattice. The lattice is called the *Bousfield lattice.* We investigate the category $\mathcal{S}$ by the Bousfield localizations and the Bousfield lattice.

In the celebrated paper [5], Miller, Ravenel and Wilson introduced the "chromatic" method to study the stable homotopy category $\mathcal{S}$ of spectra. For the nth Johnson-Wilson spectrum $E(n)$, L_n denotes the Bousfield localization functor with respect to $E(n)$. These Bousfield localizations give rise to the "chromatic tower", which is a limit system $\{L_n X\}_n$. Hopkins and Ravenel showed the chromatic convergence theorem, which implies that if X is finite, then the homotopy groups $\pi_*(X)$ is isomorphic to $\lim_n \pi_*(L_n X)$. In particular, the homotopy groups $\pi_*(S)$ of the sphere spectrum S are built from the homotopy groups $\pi_*(L_n S)$.

The algebraic K-groups of the sphere spectrum are closely related with number theory, geometric topology and so on. Bökstedt, Hsiang and Madesn defined the *cyclotomic trace map* from the algebraic K-groups of a ring spectrum X to the topological cyclic group of X, which are approximated by the TR-groups of X. Furthermore the TR-groups of the sphere spectrum are studied by the stable homotopy groups $\pi_*(S)$ and the skeleton filtration spectral sequence.

From now on, we give an overview of this thesis.

In Chapter 1, we explain the results in [2]. In the Adams-Novikov spectral sequence converging to $\pi_*(S)$, we have an element $\beta_{p/p}$ in the E_2-term $E_2^{2,2p^2(p-1)}$ which does not survive to $\pi_*(S)$. We prove that the element $\beta_{p/p}^p$ in $E_2^{2p,2p^3(p-1)}$ survives to $\pi_*(S)$ in the Adams-Novikov spectral sequence, and also give conditions to which a product of elements in the Adams-Novikov E_2-term survives to $\pi_*(S)$. Furthermore, such products are detected in $\pi_*(L_3 S)$. We investigate the third Morava stabilizer algebra for showing the result.

In Chapter 2, we look into the details of [1]. Hesselholt determined the 2-primary TR-groups of the sphere spectrum in dimensions less than 6. We extend the result to dimensions less than 10 by use of the mod 2 Adams spectral sequence and the skeleton filtration spectral sequence.

In Chapter 3, we study the Adams-Novikov spectral sequence for computing the homotopy groups of a monochromatic spectrum. The E_2-terms of the spectral sequence are the cohomology groups of a monochromatic module, to which the chromatic spectral sequence converges. In [3], we determined the first line of an E_1-term of the chromatic spectral sequence for a monochromatic module whose chromatic level is greater than 3. We look into the details of calculation for showing the result.

In the last chapter, we consider the works of [4] on a generalized Bousfield lattice. For a commutative ring R, we define the lattice $\beta(R)$, which is called the *Bousfield lattice associated to* R. In particular, the original Bousfield lattice is the Bousfield lattice associated to itself in this sense. We determine the structure of the lattice $\beta(P/I)$ for a principal ideal domain P and a nonzero ideal I of P, on which we show that the retract conjecture holds. As an application, we determine the structure of the Bousfield lattice of "harmonic" spectra, which implies that the Bousfield lattice of the category of spectra is uncountable.

Acknowledgments. I would like to thank Professor Shimomura for much valuable advice. I would also like to thank the faculty and staff of the Kochi University for their support.

Bibliography

[1] R. Kato, *The TR-groups of the sphere spectrum at the prime two*, submitted.

[2] R. Kato and K. Shimomura, *Products of Greek letter elements dug up from the third Moarava stabilizer algebra*, Algebraic and Geometric Topology 12 (2012), 951-961.

[3] R. Kato and K. Shimomura, *The first line of the Bockstein spectral sequence on a monochromatic spectrum at an odd prime*, Nagoya Mathematical Journal 207 (2012), 139-157.

[4] R. Kato, K. Shimomura and Y. Tatehara, *Generalized Bousfield lattices and a generalized retract conjecture*, Publ. Res. Inst. Math. Sci. 50 (2014), no. 3, 497-513.

[5] H. R. Miller, D. C. Ravenel and W. S. Wilson, *Periodic phenomena in the Adams-Novikov spectral sequence.* Ann. of Math. (2) 106 (1977), no. 3, 469-516.

Contents

Chapter 1

Products of Greek letter elements dug up from the third Morava stabilizer algebra

In [2], Oka and Shimomura considered the cohomology of the second Morava stabilizer algebra to study nontriviality of the products of beta elements of the stable homotopy groups of spheres. In this chapter, we use the cohomology of the third Morava stabilizer algebra to find nontrivial products of Greek letters of the stable homotopy groups of spheres: $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\langle\alpha_1,\alpha_1,\beta^p_{p/p}\rangle\gamma_t\beta_1$ and $\langle\beta_1,p,\gamma_t\rangle$ for t with $p\nmid t(t^2-1)$ for a prime number $p>5$. This is a joint work with Professor Shimomura.

1.1 Introduction

Greek letter elements are well known generators of the stable homotopy groups of spheres localized at a prime p. Studying products among these elements is an interesting subject, and studied by several authors. For example, at an odd prime p, all products of alpha elements are trivial. In [2], we used $H^*S(2)$ to study nontriviality of the product of beta elements. In this chapter, we use $H^*S(3)$ to find relations of Greek letters. The multiplicative structure of $H^*S(3)$ is given by Yamaguchi [5], but unfortunately, it has some typos. So here, our computation is based on Ravenel's.

Let $\beta_{p/p}$ be the generator of the E_2-term $E_2^{2,p^2q}(S)$ of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(S)$ of the sphere spectrum S. Hereafter, $q=2p-2$ as usual. A relation given by Toda implies that $\beta_{p/p}$ dies in the Adams-Novikov spectral sequence at a prime $p>2$. At the prime two, $\beta_{2/2}^2 = 0$ by [1, Prop. 8.22], while at the prime numbers three and five, Ravenel showed that $\beta_{p/p}^p$ survives to a homotopy element of $\pi_*(S)$ and $\alpha_1\beta_{p/p}^p = 0$ for the generator α_1 of $\pi_{q-1}(S)$. Here, we show the following

Theorem 1.1.1. *At a prime $p > 3$, $\beta_{p/p}^p$ survives to $\pi_{(p^3-1)q-2}(S)$ and $\alpha_1\beta_{p/p}^p = 0$.*

Corollary 1.1.2. *At a prime $p > 3$, the Toda bracket $\langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle$ is defined.*

We notice that at the prime 3, Ravenel showed these in [3].

Let β_1, β_2 and γ_t $(t > 0)$ be the generators of Coker J of dimensions $pq-2$, $(2p+1)q-2$ and $(tp^2+(t-1)p+t-2)q-3$, respectively.

Theorem 1.1.3. *Let $p > 5$, and t be a positive integer with $p \nmid t(t^2-1)$. Then, the elements $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\langle \alpha_1, \alpha_1, \beta_{p/p}^p \rangle \beta_1\gamma_t$ and $\langle \beta_1, p, \gamma_t \rangle$ generate subgroups of the stable homotopy groups of spheres isomorphic to $\mathbb{Z}/p$. Besides, even in the case $p|(t+1)$, $\beta_1\gamma_t$ and $\langle \beta_1, p, \gamma_t \rangle$ are generators of order p.*

Note that $\langle \beta_1, p, \gamma_t \rangle = \langle \gamma_t, p, \beta_1 \rangle$. We also notice that if $t = 1$, then $\langle \gamma_1, p, \beta_1 \rangle = 0$, while $\beta_2\gamma_1$ is non-trivial (see section five).

From here on, we assume that the prime number p is greater than three.

1.2 $H^*S(3)$ revisited

We begin with recalling some notation from Ravenel's green book [3]. Let BP denote the Brown-Peterson spectrum. Then, the pair

$$(BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

is a Hopf algebroid. Here, the degrees of v_i and t_i are $2p^i - 2$. The structure maps act as follows:

(1.2.1)

$$\begin{aligned}
\eta_R(v_1) &= v_1 + pt_1 \\
\eta_R(v_2) &\equiv v_2 + v_1 t_1^p + pt_2 \quad \bmod (p^2, v_1^p) \\
\eta_R(v_3) &\equiv v_3 + v_2 t_1^{p^2} + v_1 t_2^p + pt_3 - pv_1v_2^{p-1}t_2 \quad \bmod (p^2, v_1^2, v_2^p) \\
\Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1 \\
\Delta(t_2) &= t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{10} \\
\Delta(t_3) &\equiv t_3 \otimes 1 + t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p + 1 \otimes t_3 \quad \bmod (v_1, v_2) \\
\Delta(t_4) &\equiv t_4 \otimes 1 + t_3 \otimes t_1^{p^3} + t_2 \otimes t_2^{p^2} + t_1 \otimes t_3^p + 1 \otimes t_4 - v_3 b_{12} \quad \bmod (v_1, v_2)
\end{aligned}$$

for

(1.2.2) $$b_{1k} = \frac{1}{p} \sum_{i=1}^{p^{k+1}-1} \binom{p^{k+1}}{i} t_1^i \otimes t_1^{p^{k+1}-i}.$$

Let $K(3)_* = F_p[v_3, v_3^{-1}]$ have the BP_*-module structure given by $v_i v_3^s = v_3^s v_i = v_3^{s+1}$ if $i = 3$, and $= 0$ otherwise, and

$$\begin{aligned} \Sigma(3) &= K(3)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(3)_* \\ &= K(3)_*[t_1, t_2, \dots]/(v_3 t_i^{p^3} - v_3^{p^i} t_i : i > 0) \quad \text{(by [3, 6.1.16])} \end{aligned}$$

is the Hopf algebra with structure inherited from $BP_*(BP)$. Define the Hopf algebra $S(3)$ by $S(3) = \Sigma(3) \otimes_{K(3)_*} F_p$, where $K(3)_*$ acts on F_p by $v_3 \cdot 1 = 1$. Then,

$$S(3) = F_p[t_1, t_2, \dots]/(t_i^{p^3} - t_i : i > 0).$$

Now we abbreviate $\mathrm{Ext}_{S(3)}(F_p, F_p)$ to $H^*S(3)$.

Consider integers d_i $(= d_{3,i}$ in [3, 6.3.1])

$$d_i = \begin{cases} 0 & i \leq 0, \\ \max(i, pd_{i-3}) & i > 0. \end{cases}$$

Then, there is a unique increasing filtration of the Hopf algebroid $S(3)$ with deg $t_i^{p^j} = d_i$ for $0 \leq j < 3$.

Theorem 1.2.3. (Ravenel[3, 6.3.2]) *The associated Hopf algebra $E^0S(3)$ is isomorphic to the truncated polynomial algebra of height p on the elements $t_i^{p^j}$ for $i > 0$ and $j \in \mathbb{Z}/3$, with coproduct defined by*

$$\Delta(t_i^{p^j}) = \begin{cases} \sum_{k=0}^{i} t_k^{p^j} \otimes t_{i-k}^{p^{k+j}} & i \leq 3, \\ t_i^{p^j} \otimes 1 + 1 \otimes t_i^{p^j} + b_{i-3,j+2} & i > 3. \end{cases}$$

Let $L(3)$ be the Lie algebra without restriction with basis $x_{i,j}$ for $i > 0$ and $j \in \mathbb{Z}/3$ and bracket given by

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta_{i+j}^{l} x_{i+k,j} - \delta_{k+l}^{j} x_{i+k,l} & \text{for } i + k \leq 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_j^i = 1$ if $i \equiv j \bmod 3$ and 0 otherwise, and $L(3, k)$ the quotient of $L(3)$ obtained by setting $x_{i,j} = 0$ for $i > k$. Then, Ravenel noticed in [3, 6.3.8]:

Theorem 1.2.4. *$H^*(L(3, k))$ for $k \leq 3$ is the cohomology of the exterior complex $E(h_{i,j})$on one-dimensional generators $h_{i,j}$ with $i \leq k$ and $j \in \mathbb{Z}/3$, with coboundary*

$$d(h_{i,j}) = \sum_{s=1}^{i-1} h_{s,j} h_{i-s,s+j}.$$

From now on, we abbreviate $h_{i,j}$ to h_{ij}, and h_{1j} to h_j.

Under the above filtration, Ravenel constructed the May spectral sequences

Theorem 1.2.5. (Ravenel [3, 6.3.4, 6.3.5]) *There are spectral sequences* *(a)* $E_2 = H^*(L(3,3)) \Longrightarrow H^*(E_0S(3))$ *and*

(b) $E_2 = H^*(E_0S(3)) \Longrightarrow H^*(S(3))$.

Since these spectral sequences collapse, $H^*S(3)$ is additively isomorphic to $H^*L(3,3)$. Therefore, we have a projection

$$\pi\colon H^*S(3) \to E^0H^*S(3) = H^*(E_0S(3)) = H^*L(3,3). \tag{1.2.6}$$

Note that the Massey product $\langle h_i, h_{i+1}, h_{i+2}, h_i\rangle$ is homologous to $v_3^{(2-p)p^i} b_{i+2}$ represented by $v_3^{(2-p)p^i} b_{1,i+2}$ of (1.2.2), and π assigns the Massey product to $b_{i+2} \in H^*L(3,3)$. Ravenel determined in [3, 6.3.34] the additive structure of $H^*L(3,3)$. In particular, we have the following:

Theorem 1.2.7. $H^*L(3,3)$ *contains submodules generated by:*

$$h_1k_1\zeta_3, \quad b_0k_1\zeta_3, \quad h_0l, \quad k_0l, \quad h_0b_0b_2l \quad \textit{and} \quad h_1l.$$

Moreover $h_1l \neq h_1k_1\zeta_3$. *Here,* $l = h_2h_{21}h_{30}$, $k_i = h_{i+1}h_{2i}$ $(i = 0,1)$, $b_0 = h_1h_{32} + h_{21}h_{20} + h_{31}h_1$, $b_2 = h_0h_{31} + h_{20}h_{22} + h_{30}h_0$ *and* $\zeta_3 = h_{30} + h_{31} + h_{32}$.

Proof. In the table of [3, 6.3.34], we find the elements

$$h_0, \quad h_1, \quad k_0, \quad b_0, \quad b_2, \quad l, \quad l' = h_0h_{22}h_{31} \quad \text{and} \quad \zeta_3,$$

as well as the first element $h_1k_1\zeta_3$ of the theorem. We also have the element $h_1k_1h_{30} = h_1h_2h_{21}h_{30}$ in the table, which is the last element h_1l of the theorem. These also imply $h_1l \neq h_1k_1\zeta_3$.

The element $h_0b_0b_2l\zeta_3$ is computed as

$$\begin{aligned} & h_0h_2h_{21}h_{30}(h_1h_{32} + h_{21}h_{20} + h_{31}h_1)(h_0h_{31} + h_{20}h_{22} + h_{30}h_0)(h_{30} + h_{31} + h_{32}) \\ = \; & -2h_0h_1h_2h_{20}h_{21}h_{22}h_{30}h_{31}h_{32}. \end{aligned}$$

Therefore, $h_0b_0b_2l$ is the dual of the generator $-\frac{1}{2}\zeta_3$, and the elements $h_0b_0b_2l$ and h_0l are generators. Similarly, a computation

$$\begin{aligned} k_0ll'\zeta_3 &= h_1h_{20}h_2h_{21}h_{30}h_0h_{22}h_{31}(h_{30} + h_{31} + h_{32}) \\ &= -h_0h_1h_2h_{20}h_{21}h_{22}h_{30}h_{31}h_{32} \end{aligned}$$

shows that k_0l is the dual of the generator $-l'\zeta_3$. □

Lemma 1.2.8. *In* $H^*L(3,3)$, $h_0k_1 = 0$ *and* $k_0k_1 = 0$.

Proof. From the proof of [3, 6.3.34], we read off the relations $h_0k_1 = e_{30}h_2$ and $k_0k_1 = e_{30}g_1$ in $H^*L(3,2)$. Since e_{30} cobounds h_{30} in $H^*L(3,3)$, the lemma follows. □

1.3 Greek letter elements

Let $E_r^{s,t}(X)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to the homotopy group $\pi_{t-s}(X)$ of a spectrum X. Then the E_2-term is $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(X))$. We here consider the Ext-group $\mathrm{Ext}_{BP_*(BP)}(BP_*, M)$ for a $BP_*(BP)$-comodule M as the cohomology of the cobar complex $\Omega^*_{BP_*(BP)}M$ (*cf.* [1]). Consider a sequence $A = (a_0, a_1, \dots, a_n)$ of non-negative integers so that the sequence $p^{a_0}, v_1^{a_1}, \dots, v_n^{a_n}$ is invariant and regular. For such a sequence A, Miller, Ravenel and Wilson introduced in [1] n-th Greek letter elements $\eta^{(n)}_{s(A)}$ in the Adams-Novikov E_2-term $E_2^{n,t(A)}(S)$ by

$$\eta^{(n)}_{s(A)} \quad = \quad \delta_{A,1}\cdots\delta_{A,n}(v_n^{a_n}) \in E_2^{n,t(A)}(S) \tag{1.3.1}$$

for $v_n^{a_n} \in \mathrm{Ext}^{0,2a_n(p^n-1)}_{BP_*(BP)}(BP_*, BP_*/I(A,n))$. Here, $s(A) = a_n/a_{n-1}, a_{n-2}, \cdots, a_0$ and $t(A) = 2a_n(p^n-1) - 2\sum_{i=0}^{n-1} a_i(p^i-1)$, $I(A,k)$ denotes the ideal of BP_* generated by $p^{a_0}, v_1^{a_1}, \dots, v_{k-1}^{a_{k-1}}$, and $\delta_{A,k+1}$ is the connecting homomorphism associated to the short exact sequence

$$0 \to BP_*/I(A,k) \xrightarrow{v_k^{a_k}} BP_*/I(A,k) \to BP_*/I(A,k+1) \to 0.$$

In particular, we write $\alpha = \eta^{(1)}$, $\beta = \eta^{(2)}$ and $\gamma = \eta^{(3)}$. So far, only when $n \le 3$, we know a condition whether or not Greek letter elements survive to homotopy elements. We abbreviate $\eta^{(n)}_{s(A)}$ to $\eta^{(n)}_{a_n}$ if $A = (1, \dots, 1, a_n)$ as usual. For example, we consider β-elements defined by

$$\begin{gathered} \beta_s = \delta_{(1,1),1}(\beta'_s) \in E_2^{2,t(1,1,s)}(S) \\ \text{for } \beta'_s = \delta_{(1,1),2}(v_2^s) \in E_2^{1,t(1,1,s)}(V(0)), \text{ and} \\ \beta_{p^i/p^i} = \beta_{p^i/p^i,1} = \delta_{(1,p^i),1}\delta_{(1,p^i),2}(v_2^{p^i}) \in E_2^{2,t(1,p^i,p^i)}(S). \end{gathered} \tag{1.3.2}$$

At the prime p greater than three, we have the Smith-Toda spectrum $V(k)$ for $k = 0, 1, 2$ defined by the cofiber sequences

$$\begin{gathered} S \xrightarrow{p} S \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S, \\ \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1}V(0) \quad \text{and} \\ \Sigma^{(p+1)q}V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1}V(1). \end{gathered} \tag{1.3.3}$$

Here, $\alpha \in [V(0), V(0)]_q$ is the Adams map and $\beta \in [V(1), V(1)]_{(p+1)q}$ is the v_2-periodic element due to L. Smith. Note that the BP_*-homology of these spectra are $BP_*(V(k)) = BP_*/I_{k+1}$ for the ideal I_k of BP_* generated by v_i for $0 \le i < k$ with $v_0 = p$. We consider the Bousfield-Ravenel localization functor L_3 with respect to $v_3^{-1}BP$. The E_2-term $E_2^*(L_3V(2))$ of $L_3V(2)$ is isomorphic to $K(3)_* \otimes H^*S(3)$, whose structure is given in [3] (see also [5]), and we consider the composite

$$r\colon E_2^*(S) \xrightarrow{\iota_*} E_2^*(V(2)) \xrightarrow{\eta} E_2^*(L_3V(2)) \xrightarrow{\rho} H^*(S(3)) \xrightarrow{\pi} H^*L(3,3).$$

Here the first map is induced from the inclusion $\iota\colon S \to V(2)$ to the bottom cell, the second is from the localization map, the third is obtained by setting $v_3 = 1$ and the last is the projection (1.2.6).

Lemma 1.3.4. *The map r assigns the Greek letter elements as follows:*

$$\begin{array}{rcl} r(\alpha_1) & = & h_0, \quad r(\beta_1) \ = \ -b_0, \quad r(\beta_2) \ = \ -2k_0, \\ r(\gamma_t) & = & t(t^2-1)l - t(t-1)k_1\zeta_3 \quad \text{and} \quad r(\beta_{p/p}) \ = \ -b_1. \end{array}$$

We also have $\beta_1' = h_1 - v_1^{p-1}h_0 \in E_2^{1,pq}(V(0))$ for the generators h_i of $E_2^{1,p^iq}(V(0))$ represented by $t_1^{p^i}$.

Proof. First we consider the images of the Greek letter elements under the map $\iota_*\colon E_2^*(S) \to E_2^*(V(2))$. In the cobar complex $\Omega^*_{BP_*(BP)}BP_*$, by (1.2.1), $d(v_1) = pt_1$, $d(v_2^{p^i}) \equiv v_1^{p^i}t_1^{p^{i+1}} \bmod (p, v_1^{p^{i+1}})$ for $i \ge 0$, $d(v_2^2) \equiv 2v_1v_2t_1^p + v_1^2t_1^{2p} \bmod (p, v_1^p)$, and $d(v_3^t) \equiv tv_2v_3^{t-1}t_1^{p^2} + \binom{t}{2}v_2^2v_3^{t-2}t_1^{2p^2} \bmod (p, v_1, v_2^3)$, which imply

$$\begin{array}{rcl} \delta_{(1),1}(v_1) & = & [t_1], \quad \delta_{(1,1),2}(v_2) \ = \ [t_1^p - v_1^{p-1}t_1], \\ \delta_{(1,1),2}(v_2^2) & = & [2v_2t_1^p + v_1t_1^{2p} + v_1^2y], \quad \delta_{(1,p),2}(v_2^p) \ = \ [t_1^{p^2} - v_1^{p^2-p}t_1^p] \quad \text{and} \\ \delta_{(1,1,1),3}(v_3^t) & = & [tv_3^{t-1}t_1^{p^2} + \binom{t}{2}v_2v_3^{t-2}t_1^{2p^2} + \binom{t}{3}v_2^2v_3^{t-3}t_1^{3p^2} + v_2^3z] \ = \ \overline{\gamma}_t, \end{array}$$

for cochains $y \in \Omega^1_{BP_*(BP)}BP_*/(p)$ and $z \in \Omega^1_{BP_*(BP)}BP_*/(p, v_1)$. Here, $[x]$ denotes a cohomology class represented by a cocycle x. The first one shows $\alpha_1 = h_0$, and the second gives the last statement of the lemma. We further see that $d(t_1^{p^k}) = -pb_{1,k-1}$ for $k \ge 1$ and $d(v_k) \equiv pt_k \bmod I((2,1,1),k)$ for $k = 2,3$ by (1.2.1) in $\in \Omega^1_{BP_*(BP)}BP_*$. Moreover, $[b_{1,k}]$'s are assigned to b_k in $H^*L(3,3)$ under the projection π, and we obtain

$$\begin{array}{l} r\delta_{(1,p^{k-1}),1}(h_k - v_1^{p^k-p^{k-1}}h_{k-1}) \ = \ -b_{k-1} \quad \text{for } k = 1,2, \\ r\delta_{(1,1),1}([2v_2t_1^p + v_1t_1^{2p}]) \ = \ -2k_0, \\ \delta_{(1,1,1),2}(\overline{\gamma}_t) \ = \ [t(t-1)v_3^{t-2}t_2^p \otimes t_1^{p^2} + z] \ = \ \gamma_t' \quad \text{and} \\ r\delta_{(1,1,1),1}(\gamma_t') \ = \ t(t-1)(t-2)h_{30}k_1 + t(t-1)r\delta_{(1,1,1),1}(k_1). \end{array}$$

Here, z is a linear combination of terms in the ideal $(v_1, v_2)^2$ and of the form $v_ex \otimes y$ for $e \in \{1,2\}$ and $x, y \in \{t_i^{p^k}t_j^{p^l}, t_1^{3p^2} : i,j,k,l \in \{1,2\}\}$. Thus the relations other than $r(\gamma_t)$ follows. Note that $b_1 = h_2h_{30} + h_{22}h_{21} + h_{32}h_2$. Since $r\delta_{(1,1,1),1}(k_1) = (h_{21}h_{30} + h_{31}h_{21})h_2 - h_{21}b_1 = 3l - k_1\zeta_3$, we obtain the relation on $r(\gamma_t)$. □

Recall the cofiber sequences (1.3.3) and the v_3-periodic element $\gamma \in [V(2), V(2)]_{q_3}$ ($q_3 = (p^2+p+1)q$) due to H. Toda. Then, the Greek letter elements in homotopy are defined by

(1.3.5)
$$\alpha_t \ = \ j\alpha^t i, \quad \beta_t \ = \ j\beta_t' \quad \text{for } \beta_t' \ = \ j_1\beta^t i_1 i \quad \text{and} \quad \gamma_t \ = \ jj_1j_2\gamma^t i_2i_1i$$

for $t > 0$, and the Greek elements in the E_2-term survives to the same named one in homotopy by the Geometric Boundary Theorem (*cf.* [3]).

Proof of Theorem 3.1.10. We begin with noticing that the element b_i in $H^*L(3,3)$ is the image of the Massey product $\langle h_i, h_{i+1}, h_{i+2}, h_i\rangle$ under π, which is homologous to b_i represented by $b_{1,i}$ in (1.2.2). We further note that the Toda brackets $\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle$ and $\langle\beta_1, p, \gamma_t\rangle$ are detected by $\alpha_1 b_2$ and $h_1\gamma_t$ of $E_2^*(S)$, respectively. Indeed, in the first bracket, $d_{2p-1}(b_2) = \alpha_1\beta^p_{p/p}$ by Corollary 1.4.4 below, and in the second bracket, $\langle\beta_1, p, \gamma_t\rangle = j\langle\beta'_1, p, \gamma_t\rangle$. Under the condition on t, Lemmas 1.3.4, 1.2.7 and 1.2.8 imply that each element of $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\alpha_1 b_2\gamma_t\beta_1$ and $h_1\gamma_t$, as well as $\beta_1\gamma_t$, generates a submodule isomorphic to $\mathbb{Z}/p$ of the E_2-term $E_2^*(S)$. These are, of course, permanent cycles, and nothing kills them in the Adams-Novikov spectral sequence since each element has a filtration degree less than $2p-1$. □

1.4 $\beta^p_{p/p}$ in the homotopy of spheres

Let X and $\overline{X}$ be the $(p-1)q$- and $(p-2)q$-skeletons of the Brown-Peterson spectrum BP. Then, we have the cofiber sequences

(1.4.1) $$S \xrightarrow{\iota} X \xrightarrow{\kappa} \Sigma^q\overline{X} \xrightarrow{\lambda} S^1 \quad\text{and}\quad \overline{X} \xrightarrow{\iota'} X \xrightarrow{\kappa'} S^{(p-1)q} \xrightarrow{\lambda'} \Sigma\overline{X}.$$

Then,

$$BP_*(X) = BP_*[x]/(x^p) \quad\text{and}\quad BP_*(\overline{X}) = BP_*[x]/(x^{p-1})$$

as subcomodules of $BP_*(BP)$, where x corresponds to t_1. From [3, Chap.7], we read off the following:

(1.4.2) $b_1^p = 0 \in E_2^{2p,p^3q}(X)$, *which implies*

$$E_2^{2s+e,tq}(X) = 0 \quad \textit{if } s \geq p \textit{ and } t < (s-1)p^2 + (s+1+e)p.$$

Lemma 1.4.3. $b_0\colon E_2^{2s+e,tq}(S) \to E_2^{2s+2+e,(t+p)q}(S)$ *is monomorphic if* $s \geq p$ *and* $t \leq (s-1)p^2 + (s+e)p$.

Proof. Note that $b_0 = \lambda\lambda'$, and the lemma follows from (1.4.2) and the exact sequences

$$E_2^{2s+e,(t+p-1)q}(X) \xrightarrow{\kappa'} E_2^{2s+e,tq}(S) \xrightarrow{\lambda'} E_2^{2s+1+e,(t+p-1)q}(\overline{X})$$
$$E_2^{2s+e+1,(t+p)q}(X) \xrightarrow{r} E_2^{2s+e+1,(t+p-1)q}(\overline{X}) \xrightarrow{\lambda} E_2^{2s+2+e,(t+p)q}(S)$$

induced from the cofiber sequences in (1.4.1). □

Ravenel showed that $d_{2p-1}(\beta_{p^2/p^2}) \equiv \alpha_1\beta^p_{p/p}$ mod Ker β_1^p in the Adams-Novikov spectral sequence for $\pi_*(S)$ (*cf.* [3, 6.4.1]). Here, the mapping β_1^p on $E_2^{2p+1,(p^3+1)q}(S)$ is a monomorphism by Lemma 1.4.3:

Corollary 1.4.4. *In the Adams-Novikov spectral sequence for* $\pi_*(S)$, $d_{2p-1}(\beta_{p^2/p^2}) = \alpha_1\beta^p_{p/p} \in E^{2p+1,(p^3+1)q}_{2p-1}(S) = E^{2p+1,(p^3+1)q}_2(S)$.

Proof of Theorem 1.1.1. Consider the first cofiber sequence in (1.4.1). Since the Adams-Novikov E_2-term $E_2^{sq+3,(p^3+s)q}(X)$ vanishes for $s > 0$ by (1.4.2), the element $\iota_*(\beta_{p^2/p^2}) \in E_2^{2,p^3q}(X)$ survives to a homotopy element ${}^X\beta_{p^2/p^2} \in \pi_*(X)$. In general, we see that

(1.4.5) *Let* $\bar\iota\colon S \to \overline{X}$ *denote the inclusion to the bottom cell. Then,* $\lambda_*\bar\iota(x) = \alpha_1 x$ *for* $x \in E_2^*(S)$.

Put $\overline{\beta}_{p/p} = \bar\iota_*(\beta_{p/p}) \in E_2^{2,p^2q}(\overline{X})$, and we see that $\lambda_*(\overline{\beta}^p_{p/p}) = \alpha_1\beta^p_{p/p}$, and so we see that $\overline{\beta}^p_{p/p}$ detects an essential homotopy element $\kappa_*({}^X\beta_{p^2/p^2}) \in \pi_*(\overline{X})$ by Corollary 1.4.4, which we also denote by $\overline{\beta}^p_{p/p}$.

Now turn to the second cofiber sequence in (1.4.1). The relation $b_1^p = 0$ of (1.4.2) yields a cochain $y = \sum_{i=0}^{p-1} x^i y_i \in \Omega^{2p-1}BP_*(X)$ such that $d(y) = b_1^p$, where $y_i \in \Omega^{2p-1}BP_*$. It follows that $d(\overline{y}) = b_1^p - d(x^{p-1})y_{p-1} \in \Omega^{2p}BP_*(\overline{X})$ for $\overline{y} = \sum_{i=0}^{p-2} x^i y_i \in \Omega^{2p-1}BP_*(\overline{X})$. In particular $d(y_{p-1}) = 0 \in \Omega^{2p-1}BP_*$ and $d(y_{p-2}) = (1-p)t_1 \otimes y_{p-1}$. By definition, these imply $\lambda'_*(y_{p-1}) = b_1^p$. Consider the exact sequence obtained by applying the homotopy groups to the second cofiber sequence. Then, $\iota'_*(\overline{\beta}^p_{p/p}) = 0$ by (1.4.2), and so $\overline{\beta}^p_{p/p}$ must be pulled back to an element $\xi \in \pi_*(S)$ detected by y_{p-1}. Since $b_0 = \lambda\lambda'$, $b_0 y_{p-1} = h_0 b_1^p$, and $\langle h_0, \dots, h_0\rangle y_{p-1} = h_0\langle h_0, \dots, h_0, y_{p-1}\rangle$, we see that

$$b_1^p \equiv \langle h_0, \dots, h_0, y_{p-1}\rangle \not\equiv 0 \in E_2^{2p,p^3q}(S) \mod \ker h_0.$$

Put $b_1^p = \langle h_0, \dots, h_0, y_{p-1}\rangle + c$ for $c \in \ker h_0 \subset E_2^{2p,p^3q}(S)$. Then, $b_1^p - c$ survives to $\beta^p_{p/p} \in \pi_*(S)$.

The element $\alpha_1\beta^p_{p/p}$ is detected by $h_0(b_1^p - c) = h_0 b_1^p$ in the Adams-Novikov E_2-term, which is killed by b_2 by Corollary 1.4.4. □

1.5 Remarks

1.5.1 A relation on Toda bracket

The relation $\langle\beta_s, p, \gamma_t\rangle = \langle\gamma_t, p, \beta_s\rangle$ follows immediately from results of Toda: By definition, $\langle\beta_s, p, \gamma_t\rangle = j\beta_{(s)}\gamma_{(t)}i$ and $\langle\gamma_t, p, \beta_s\rangle = j\gamma_{(t)}\beta_{(s)}i$ for $\beta_{(s)} = j_1\beta^s i_1$ and $\gamma_{(t)} = j_1 j_2\gamma^t i_2 i_1$. Since $V(2)$ and $V(3)$ are $V(0)$-module spectra, $\theta(\beta) = 0$ and $\theta(\gamma) = 0$ by [4, Lemma 2.3]. Similarly, $\theta(i_k) = 0$ and $\theta(j_k) = 0$ for $k = 1, 2$. Therefore, [4, Lemma 2.2] implies $\theta(\beta_{(s)}) = 0$ and $\theta(\gamma_{(t)}) = 0$. Therefore, $\beta_{(s)}\gamma_{(t)} = \gamma_{(t)}\beta_{(s)}$ by [4, Cor. 2.7] as desired.

1.5.2 On the action of γ_1

Note that $\gamma_1 = \alpha_1\beta_{p-1}$. Then, $\alpha_1\gamma_1 = \alpha_1^2\beta_{p-1} = 0$, $\langle\alpha_1, \alpha_1, \beta_{p/p}^p\rangle\beta_1\gamma_1 = -\alpha_1\langle\alpha_1, \alpha_1, \beta_{p/p}^p\rangle\beta_1\beta_{p-1} = -\langle\alpha_1, \alpha_1, \alpha_1\rangle\beta_{p/p}^p\beta_1\beta_{p-1} = 0$ since $\langle\alpha_1, \alpha_1, \alpha_1\rangle = 0$, and $\langle\gamma_1, p, \beta_1\rangle = \beta_{p-1}\langle\alpha_1, p, \beta_1\rangle = \beta_{p-1}j\underline{\alpha j_1}\beta i_1 i = 0$.

For $t \geq 2$,

$$\begin{aligned}
\beta_t &= \delta_{(1,1),1}\delta_{(1,1),2}(v_2^t) = \delta_{(1,1),1}([tv_2^{t-1}t_1^p + \tbinom{t}{2}v_1v_2^{t-2}t_1^{2p} + v_1^2x]) \\
&\equiv [t(t-1)v_2^{t-2}t_2 \otimes t_1^p - tv_2^{t-1}b_0 + \tbinom{t}{2}v_2^{t-2}t_1 \otimes t_1^{2p}] \mod (p, v_1) \\
&\equiv t(t-1)v_2^{t-2}k_0 - tv_2^{t-1}b_0 \mod (p, v_1)
\end{aligned}$$

and $\alpha_1\beta_2\beta_{p-1} \in E_2^5(S^0)$ is projected to $h_0(2k_0 - 2v_2b_0)(2v_2^{p-3}k_0 + v_2^{p-2}b_0) = -2v_2^{p-2}h_0k_0b_0 - 2h_0v_2^{p-1}b_0^2$ in $E_2^5(V(2))$ under the induced map i_* from the inclusion $i\colon S^0 \to V(2)$ to the bottom cell. Here, $k_0 = [t_2 \otimes t_1^p + \frac{1}{2}t_1 \otimes t_1^{2p}]$. Then, this element is detected by $-2v_2^{p-2}k_0 \in E_1^3 = E_2^{2,(p^2+p-1)q}(X \wedge V(2))$ in the small descent spectral sequence. The killer of this element, if any, stays in the E_1-terms $E_1^2 = E_2^{2,(p^2+p)q}(X \wedge V(2))$, $E_1^1 = E_2^{3,(p^2+2p-1)q}(X \wedge V(2))$ and $E_1^0 = E_2^{4,(p^2+2p)q}(X \wedge V(2))$. These are zero, and we see that the product is not zero.

Bibliography

[1] H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), 469–516.

[2] S. Oka and K. Shimomura, *On product of the β-elements in the stable homotopy of spheres*, Hiroshima Math. J. **12** (1982), 611–626.

[3] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.

[4] H. Toda, *Algebra of stable homotopy of Z_p-spaces and applications*, J. Math. Kyoto Univ., **11** (1971), 197–251.

[5] A. Yamaguchi, *The structure of the cohomology of Morava stabilizer algebra* $S(3)$, Osaka J. Math **29** (1992), 347–359.

Chapter 2

The TR-groups of the sphere spectrum at the prime two

For the multiplicative group S^1, the circle, we have the topological Hochschild S^1-spectrum $T(\mathbb{S})$ of the sphere spectrum $\mathbb{S}$. For the finite cyclic group C_r ($\subset S^1$) of order r, the *TR-groups of $\mathbb{S}$ at* 2 are defined by the equivariant homotopy groups $TR^n_k(\mathbb{S};2) = [S^k \wedge (S^1/C_{2^{n-1}})_+, T(\mathbb{S})]_{S^1}$ for $k \geq 0$ and $n \geq 1$. By the "trace method", the groups are closely related with the algebraic K-groups of $\mathbb{S}$. In [1], Hesselholt determined the TR-groups for $0 \leq k \leq 5$, in order to obtain the homotopy groups of the topological Whitehead spectrum of the circle in dimensions less than 4. In this chapter, we extend his result for the TR-groups to $k \leq 9$ by the mod 2 Adams spectral sequence as well as the Atiyah-Hirzebruch spectral sequence.

2.1 Introduction

Throughout this chapter, we fix a prime $p = 2$ and denote by C_r the finite cyclic subgroup of the circle S^1 of order r. Let $T(X)$ denote the topological Hochschild homology spectrum of a ring spectrum X. Since $T(X)$ is an S^1-spectrum, we define the *TR-spectrum $TR^n(X;2)$ of level n* as the fixed point spectrum $T(X)^{C_{2^{n-1}}}$ for $n \geq 1$. The spectrum $TR(X;2)$ is given by

$$TR(X;2) = \mathrm{holim}_n TR^n(X;2),$$

the homotopy limit of the system $\{R\colon TR^n(X;2) \to TR^{n-1}(X;2)\}_n$ of the restriction maps. The Frobenius maps $F\colon TR^n(X;2) \to TR^{n-1}(X;2)$ induce a map $F\colon TR(X;2) \to TR(X;2)$, and $TC(X;2)$ is a spectrum fitting in the cofiber sequence

$$TC(X;2) \xrightarrow{i} TR(X;2) \xrightarrow{id-F} TR(X;2) \xrightarrow{\partial} \Sigma TC(X;2).$$

Consider the algebraic K-theory spectrum $K(X)$ of a ring spectrum X, and the cyclotomic trace map $trc\colon K(X) \to TC(X;2)$. The "trace method" is to study $K(X)$ through the composite

$$tr_n\colon K(X) \xrightarrow{trc} TC(X;2) \xrightarrow{i} TR(X;2) \to TR^n(X;2).$$

We call the homotopy groups $TR^n_*(X;2) = \pi_*(TR^n(X;2))$ the (*2-primary*) *TR-groups of X of level n*.

Let $\mathbb{S}$ denote the sphere spectrum localized at the prime two. In this chapter, we consider the TR-groups $TR^n_*(\mathbb{S};2)$. We have the Segal-tom Dieck splitting $TR^n_*(\mathbb{S};2) \cong \pi^S_*((BC_{2^{n-1}})_+) \oplus TR^{n-1}_*(\mathbb{S};2)$ ([1, p. 137, p. 148, p. 155]), where $BC_{2^{n-1}}$ denotes the classifying space of $C_{2^{n-1}}$. By definition, $TR^1_*(\mathbb{S};2) = \pi_*(T(\mathbb{S}))$, which is isomorphic to $\pi_*(\mathbb{S})$ ([1, p. 147]). These show an isomorphism

$$TR^n_*(\mathbb{S};2) \cong \pi_*(\mathbb{S}) \oplus \textstyle\bigoplus_{1\le k<n} \pi^S_*((BC_{2^k})_+). \tag{2.1.1}$$

Hesselholt studied the Atiyah-Hirzebruch spectral sequence

$$E^2_{s,t}(n) = H_s(C_{2^n}, \pi_t(\mathbb{S})) \Rightarrow \pi^S_*((BC_{2^n})_+) \cong \pi_*(\mathbb{S}) \oplus \pi^S_*(BC_{2^n}), \tag{2.1.2}$$

which is called the skeleton spectral sequence in [1, p. 148], to show the following theorem.

Theorem 2.1.3 (Hesselholt [1, Theorem 11])**.** *The TR-groups $TR^n_k(\mathbb{S};2)$ for $k \le 5$ are given by*

$$\begin{aligned}
TR^n_0(\mathbb{S};2) &\cong \mathbb{Z}_{(2)}^{\oplus n},\\
TR^n_1(\mathbb{S};2) &\cong \mathbb{Z}/2^{\oplus n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2^s,\\
TR^n_2(\mathbb{S};2) &\cong \mathbb{Z}/2^{\oplus n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2,\\
TR^n_3(\mathbb{S};2) &\cong \mathbb{Z}/8^{\oplus n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2^{\max\{3,s+1\}} \oplus \bigoplus_{2\le s<n} \mathbb{Z}/2,\\
TR^n_4(\mathbb{S};2) &\cong \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2^{\min\{3,s\}},\\
TR^n_5(\mathbb{S};2) &\cong \textstyle\bigoplus_{2\le s<n} \mathbb{Z}/2^s \oplus \bigoplus_{3\le s<n} \mathbb{Z}/2.
\end{aligned}$$

Liulevicius determined the stable homotopy groups $\pi^S_k(BC_2)$ for $k \le 9$ ([3, Theorem II.6]). We consider $\pi^S_k(BC_{2^n})$ for $n > 1$ and $k \le 9$ in this chapter. In section 2, we determine the stable homotopy group $\pi^S_6(BC_{2^n})$ by the Atiyah-Hirzebruch spectral sequence, and in section 3, we determine the stable homotopy groups $\pi^S_*(BC_{2^n})$ in dimensions 7, 8 and 9 by the mod 2 Adams spectral sequence as well as the results in section 2. The following theorem summarizes Corollary 2.2.10 and Propositions 2.3.12, 2.3.14 and 2.3.16.

Theorem 2.1.4. *The TR-groups $TR^n_k(\mathbb{S};2)$ for $6 \le k \le 9$ are given by*

$$\begin{aligned}
TR^n_6(\mathbb{S};2) &\cong \mathbb{Z}/2^{\oplus n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2,\\
TR^n_7(\mathbb{S};2) &\cong \mathbb{Z}/16^{\oplus n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2 \oplus \bigoplus_{1\le s<n} \mathbb{Z}/2^{\max\{4,s+2\}} \oplus \bigoplus_{2\le s<n} \mathbb{Z}/2,\\
TR^n_8(\mathbb{S};2) &\cong \mathbb{Z}/2^{\oplus 2n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2^{\min\{4,s\}} \oplus \bigoplus_{1\le s<n} \mathbb{Z}/2^{\oplus 2},\\
TR^n_9(\mathbb{S};2) &\cong \mathbb{Z}/2^{\oplus 3n} \oplus \textstyle\bigoplus_{1\le s<n} \mathbb{Z}/2^{\oplus 3} \oplus \bigoplus_{1\le s<n} \mathbb{Z}/2^{\min\{4,s\}} \oplus \bigoplus_{2\le s<n} \mathbb{Z}/2^{s-1}.
\end{aligned}$$

2.2 The Atiyah-Hirzebruch spectral sequences

In this section, $E^r_{s,t}(n)$ denotes an E^r-term of the Atiyah-Hirzebruch spectral sequence (2.1.2), and $E^*(n)$ stands for the spectral sequence. Since the C_{2^n}-action on the homotopy groups $\pi_*(\mathbb{S})$ is trivial ([1, p. 145]), the standard resolution gives rise to isomorphisms

$$(2.2.1) \qquad E^2_{s,t}(n) = H_s(C_{2^n}, \pi_t(\mathbb{S})) \cong \begin{cases} \pi_t(\mathbb{S}) & s = 0, \\ \pi_t(\mathbb{S})/(2^n) & s : odd > 0, \\ \pi_t(\mathbb{S})[2^n] & s : even > 0, \end{cases}$$

of groups, where $\pi_t(\mathbb{S})[2^n]$ denotes the kernel of $\pi_t(\mathbb{S}) \xrightarrow{2^n} \pi_t(\mathbb{S})$.

Theorem 2.2.2 (*cf.* Toda [5, p. 189–190])**.** *The homotopy groups $\pi_k(\mathbb{S})$ for $k \leq 10$ are given by*

k	0	1	2	3	4	5	6	7	8	9	10
$\pi_k(\mathbb{S})$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2$
gen.	ι	η	η^2	ν			ν^2	σ	$\eta\sigma,\ \varepsilon$	$\eta\varepsilon,\ \mu,\ \nu^3$	$\eta\mu$

The generators satisfy the relations $\eta^3 = 4\nu$, $\eta^2\sigma = \eta\varepsilon + \nu^3$ and $\nu\sigma = 0$.

We notice that the spectral sequence (2.1.2) splits into the direct sum of two spectral sequences

$$E^2_{0,*}(n) \Rightarrow \pi_*(\mathbb{S}) \quad \text{and} \quad \bigoplus\nolimits_{s\geq 1} E^2_{s,*}(n) \Rightarrow \pi^S_*(BC_{2^n})$$

([1, p. 148]). We study the latter spectral sequence.

First we consider the case for $n = 1$. By (2.2.1) and Theorem 2.2.2, the E^2-terms $E^2_{s,t}(1)$ for $s \geq 1$ and $s + t \leq 10$ are given by

s										
10										0
9									$\mathbb{Z}/2$	$\mathbb{Z}/2$
8								0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7							$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
6						0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$4\mathbb{Z}/8$	0
5					$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0
4				0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$4\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$
3			$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
2		0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\boxed{4\mathbb{Z}/8}$	0	0	$\mathbb{Z}/2$	$8\mathbb{Z}/16$	$\mathbb{Z}/2^{\oplus 2}$
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 3}$
	1	2	3	4	5	6	7	8	9	10 $s+t$

Hereafter $2^a\mathbb{Z}/2^b$ denotes the subgroup of $\mathbb{Z}/2^b$ generated by 2^a, which is isomorphic to $\mathbb{Z}/2^{b-a}$ if $a < b$, and zero otherwise. For example, in the above chart, the boxed $4\mathbb{Z}/8$ at $(s,t) = (2,3)$ is the subgroup of $\mathbb{Z}/8 \cdot \nu$ generated by 4ν.

We deduce

$$(E^2_{s,t}(n) \xrightarrow{d^2} E^2_{s-2,t+1}(n)) = \begin{cases} \times\eta & 4 \leq s \equiv 0, 1 \bmod (4), \\ 0 & \textit{otherwise}, \end{cases} \tag{2.2.3}$$

from [1, p. 148]. This implies that the E^3-terms have a periodicity:

2.2.4 *The E^3-term $E^3_{s,t}(n)$ is isomorphic to $E^3_{s+4,t}(n)$ if $s \geq 2$.*

We obtain the E^3-terms $E^3_{s,t}(1)$ for $s \geq 1$ and $s+t \leq 9$ as follows by (2.2.3) and (2.2.4).

s										
9									0	
8								0	0	
7							$\mathbb{Z}/2$	0	0	
6						0	$\mathbb{Z}/2$	0	0	
5					0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	
4				0	0	0	$4\mathbb{Z}/8$	0	0	
3			$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	
2		0	$\mathbb{Z}/2$	0	0	0	0	$\mathbb{Z}/2$	$8\mathbb{Z}/16$	
1	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{\oplus 2}$	
	1	2	3	4	5	6	7	8	9	$s+t$

Theorem 2.2.5 (Liulevicius [3, Theorem II.6])**.** *The stable homotopy groups of $BC_2 = \mathbb{R}P^\infty$, the infinite real projective space, in dimensions less than 10 are given by*

k	1	2	3	4	5	6	7	8	9
$\pi^S_k(\mathbb{R}P^\infty)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/16 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 4}$

Corollary 2.2.6. *The spectral sequence $E^*(1)$ collapses at E^3 for $s+t \leq 9$.*

We turn to the case for $n \geq 2$. By (2.2.3) and (2.2.4), we have the following chart of E^3-terms of $E^*(n)$ for $s \geq 1$ and $s+t \leq 10$:

s											
10										0	
9									$2\mathbb{Z}/2^n$	0	
8								0	0	0	
7							$\mathbb{Z}/2^n$	0	0	$\mathbb{Z}/4$	
6						0	$\boxed{\mathbb{Z}/2}$	0	$\widetilde{K}_{3,n}$	0	
5					$2\mathbb{Z}/2^n$	0	Z_n	$C_{3,n}$	0	0	
4				0	0	0	$K_{3,n}$	0	0	$\boxed{\mathbb{Z}/2}$	
3			$\mathbb{Z}/2^n$	0	0	$\boxed{\mathbb{Z}/4}$	0	0	$\mathbb{Z}/2$	$C_{7,n}$	
2		0	$\mathbb{Z}/2$	0	$\widetilde{K}_{3,n}$	0	0	$\mathbb{Z}/2$	$K_{7,n}$	$\widetilde{K}_{8,n}$	
1	$\mathbb{Z}/2^n$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$C_{3,n}$	0	0	$\mathbb{Z}/2$	$C_{7,n}$	$\boxed{\mathbb{Z}/2^{\oplus 2}}$	$\mathbb{Z}/2^{\oplus 3}$	
	1	2	3	4	5	6	7	8	9	10	$s+t$

Here, $K_{t,n} = \pi_t(\mathbb{S})[2^n]$, $\widetilde{K}_{t,n} = K_{t,n}/(\eta)$, $C_{t,n} = \pi_t(\mathbb{S})/2^n$ and $Z_n = \ker(C_{2,n} \xrightarrow{\eta} C_{3,n})$, whose structures are:

$$K_{3,n} \cong 2^{\max\{3-n,0\}}\mathbb{Z}/8, \quad K_{7,n} \cong 2^{\max\{4-n,0\}}\mathbb{Z}/16, \quad \widetilde{K}_{3,n} \cong 2^{\max\{3-n,0\}}\mathbb{Z}/4,$$
$$\widetilde{K}_{8,n} \cong \mathbb{Z}/2 \text{ except for } \widetilde{K}_{8,2} \cong \widetilde{K}_{8,3} \cong \mathbb{Z}/2^{\oplus 2}, \quad C_{3,n} \cong \mathbb{Z}/2^{\min\{n,3\}},$$
$$C_{7,n} \cong \mathbb{Z}/2^{\min\{n,4\}} \quad \text{and} \quad Z_n = 0 \text{ except for } Z_2 \cong \mathbb{Z}/2.$$

Lemma 2.2.7 ([1, p. 145, Lemma 6, p. 148])**.** *The Verschiebung map $V\colon \pi_*^S((BC_{2^{n-1}})_+) \to \pi_*^S((BC_{2^n})_+)$ induces a map $V\colon E^*(n-1) \to E^*(n)$ of spectral sequences. Let $\{x\}_n$ denote an element of $E^2_{s,t}(n)$ represented by $x \in \pi_t(\mathbb{S})$. If s is even, then $V(\{x\}_{n-1}) = \{x\}_n$ for the map $V\colon E^2_{s,t}(n-1) \to E^2_{s,t}(n)$ of the E_2-terms.*

Since the differentials $E^3_{6,1}(1) \xrightarrow{d^3} E^3_{3,3}(1)$ and $E^3_{4,6}(1) \xrightarrow{d^3} E^3_{1,8}(1)$ are trivial by Corollary 2.2.6, the differentials $E^3_{6,1}(n) \xrightarrow{d^3} E^3_{3,3}(n)$ and $E^3_{4,6}(n) \xrightarrow{d^3} E^3_{1,8}(n)$ for $n \geq 2$ are trivial by Lemma 2.2.7.

Recall [1, Lemma 8] that

$$\text{2.2.8} \quad (E^4_{s,t}(n) \xrightarrow{d^4} E^4_{s-4,t+3}(n)) = \begin{cases} \times\nu & 4 < s \equiv 0,1,2,3,8,9,10,11 \bmod (16), \\ \times 2\nu & 4 < s \equiv 6,7,12,13 \bmod (16), \\ 0 & \textit{otherwise}, \end{cases}$$

for $n \geq 1$. We then obtain the following chart of the E^5-terms for $n \geq 2$, except for the underlined group $E^5_{7,3}(n)$.

s										
10										0
9									$\boxed{8\mathbb{Z}/2^n}$	0
8								0	0	0
7							$2\mathbb{Z}/2^n$	0	0	$\underline{0}$
6						0	$\mathbb{Z}/2$	0	$\boxed{\widetilde{K}_{3,n}}$	0
5					$2\mathbb{Z}/2^n$	0	Z_n	$\mathbb{Z}/2$	0	0
4				0	0	0	$K_{3,n}$	0	0	?
3			$\mathbb{Z}/2^n$	0	0	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	?
2		0	$\mathbb{Z}/2$	0	$\widetilde{K}_{3,n}$	0	0	$\boxed{\mathbb{Z}/2}$	$K_{7,n}$	?
1	$\mathbb{Z}/2^n$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$C_{3,n}$	0	0	$\mathbb{Z}/2$	$\boxed{C_{7,n}}$	$\mathbb{Z}/2^{\oplus 2}$	?
	1	2	3	4	5	6	7	8	9	10 $s+t$

By (2.2.1) and Theorem 2.2.2, we see that $E^4_{11,0}(n) = \mathbb{Z}/2^n \cdot \iota$, and that $E^4_{7,3}(n)$ is a quotient of $E^3_{7,3}(n) = \mathbb{Z}/4 \cdot \nu$. Thus, we deduce from (2.2.8) that the group $E^5_{7,3}(n)$ is zero.

Lemma 2.2.9. *On $E^r_{s,t}(n)$ for $n \geq 2$ and $r \geq 5$, the only possibly nonzero differentials are $E^5_{6,3}(n) \xrightarrow{d^5} E^5_{1,7}(n), E^7_{9,0}(n) \xrightarrow{d^7} E^7_{2,6}(n)$ and $E^8_{9,0}(n) \xrightarrow{d^8} E^8_{1,7}(n)$ for $s+t \leq 10$.*

Corollary 2.2.10. *For $n \geq 2$, the stable homotopy groups $\pi^S_*(BC_{2^n})$ in dimensions from 6 to 9 satisfy the following relations*:

$$\begin{array}{rcl} \pi^S_6(BC_{2^n}) & \cong & \mathbb{Z}/2, \\ |\pi^S_7(BC_{2^n})| & = & 2^{n+4}, \\ |\pi^S_8(BC_{2^n})| & \leq & 2^{\min\{n+2,6\}}, \\ |\pi^S_9(BC_{2^n})| & \leq & 2^{\min\{2n+2,n+6\}}. \end{array}$$

2.3 The mod 2 Adams spectral sequence

In this section, we consider the mod 2 Adams spectral sequence

$$E^{s,t}_2(X) = \mathrm{Ext}^{s,t}_{\mathcal{A}}(\widetilde{H}^*(X), \mathbb{Z}/2) \Rightarrow \pi^S_{t-s}(X)$$

for a space X. Here $\widetilde{H}^*(X)$ denotes a reduced cohomology of X with coefficients $\mathbb{Z}/2$, and $\mathcal{A}$ denotes the Steenrod algebra. We assume that $n \geq 2$, and determine the stable homotopy groups $\pi^S_*(BC_{2^n})$ in dimensions less than 10 by the mod 2 Adams spectral sequence for BC_{2^n}.

Proposition 2.3.1. *The E_2-term $E_2^{*,*}(BC_{2^n})$ is isomorphic to $xE_2^{*,*}(S^0) \oplus E_2^{*,*}(\mathbb{C}P^\infty) \oplus xE_2^{*,*}(\mathbb{C}P^\infty)$ as a graded $E_2^{*,*}(S^0)$-module for a generator $x \in E_2^{1,0}(BC_{2^n})$. Here S^0 and $\mathbb{C}P^\infty$ denote the 0-dimensional sphere and the infinite complex projective space, respectively.*

Proof. We claim that there exists a generator $x \in \widetilde{H}^1(BC_{2^n})$ such that

$$\widetilde{H}^*(BC_{2^n}) \cong \mathbb{Z}/2 \cdot x \oplus \widetilde{H}^*(\mathbb{C}P^\infty) \oplus x\widetilde{H}^*(\mathbb{C}P^\infty) \tag{2.3.2}$$

as a graded $\mathcal{A}$-algebra. Indeed, the unreduced cohomology $H^*(BC_{2^n}, \mathbb{Z}/2)$ is isomorphic to the group cohomology $H^*(C_{2^n}, \mathbb{Z}/2) \cong E(x) \otimes P(y)$ with $|x| = 1$ and $|y| = 2$. Here $E(-)$ and $P(-)$ denote the exterior and the polynomial algebras, respectively. Furthermore, we see that the action of $\mathcal{A}$ on the generators x and y is trivial except for $Sq^2(y) = y^2$ by the fundamental properties of the Steenrod squares, other than $Sq^1(y) = 0$. Note that Sq^1 fits in the exact sequence

$$\begin{aligned} H^1(BC_{2^n}, \mathbb{Z}/2) &\xrightarrow{Sq^1} H^2(BC_{2^n}, \mathbb{Z}/2) \to H^2(BC_{2^n}, \mathbb{Z}/4) \\ &\to H^2(BC_{2^n}, \mathbb{Z}/2) \xrightarrow{Sq^1} H^3(BC_{2^n}, \mathbb{Z}/2) \end{aligned}$$

associated to the short exact sequence $0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0$. In the exact sequence, $H^2(BC_{2^n}, \mathbb{Z}/2^i) \cong \mathbb{Z}/2^i$ by the standard resolution. The first Sq^1 is zero, and so is the second Sq^1 as desired. We note that $\widetilde{H}^*(S^0) \cong \mathbb{Z}/2$ and $\widetilde{H}^*(\mathbb{C}P^\infty) \cong \overline{P}(y)$ as graded $\mathcal{A}$-algebras for the augmented ideal $\overline{P}(y)$ of $P(y)$. Thus, the claim (2.3.2) is verified and hence the proposition follows. $\square$

The E_2-terms $E_2^{s,t}(S^0)$ are well known as follows ([4, Theorem 3.2.11]):

s

$\vdots$	$\vdots$									
5	h_0^5									Ph_1
4	h_0^4							$h_0^3h_3$		h_1c_0
3	h_0^3			$h_1^3 = h_0^2h_2$				$h_0^2h_2$	c_0	$h_2^3 = h_1^2h_3$
2	h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3	
1	h_0	h_1		h_2				h_3		
0	1									
	0	1	2	3	4	5	6	7	8	9

$t - s$

The generators satisfy the relations:

$$\begin{gathered} h_ih_{i+1} = 0 \text{ for } i \geq 0, \quad h_1^3 = h_0^2h_2, \quad h_0h_2^2 = 0, \quad h_2^3 = h_1^2h_3, \\ h_0^4h_3 = 0, \quad h_0c_0 = 0, \quad h_1^2c_0 = 0 \quad \text{and} \quad h_0Ph_1 = 0. \end{gathered} \tag{2.3.3}$$

We see the following fact immediately.

2.3.4 *The* mod 2 *Adams spectral sequence for S^0 collapses at E_2 for $t - s < 10$.*

The E_2-terms $E_2^{s,t}(\mathbb{C}P^\infty)$ are determined in [3, Prop. II.3] as follows:

s											
$\vdots$			$\vdots$		$\vdots$		$\vdots$		$\vdots$		$\vdots$
5			$h_0^5e_2$		$h_0^4e_4$		$h_0^5e_6$		$h_0^2e_8$		$h_0^3e_{10}$
4			$h_0^4e_2$		$h_0^3e_4$		$h_0^4e_6$		h_0e_8	$h_0^3h_3e_2$	$h_0^2e_{10}$
3			$h_0^3e_2$		$h_0^2e_4$		$h_0^3e_6$		e_8	$h_0^2h_3e_2$	h_0e_{10}
2			$h_0^2e_2$		h_0e_4	$h_0h_2e_2$	$h_0^2e_6$		$h_1^2e_6$	$h_0h_3e_2$	e_{10}
1			h_0e_2		e_4	h_2e_2	h_0e_6	h_1e_6		h_3e_2	
0			e_2				e_6				
	0	1	2	3	4	5	6	7	8	9	10
											$t-s$

Remark 2.3.5. In [3], the generators $h_0, h_i (i > 0), e_2, e_4, e_6, e_8$ and e_{10} here are denoted by $g_0, h_{i-1}, e_{0,2}, e_{1,5}, e_{0,6}, e_{3,11}$ and $e_{2,12}$, respectively.

Therefore, we obtain the following chart of $E_2^{*,*}(BC_{2^n})$ by Proposition 2.3.1.

s											
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
6		xh_0^6	$h_0^6e_2$	$xh_0^6e_2$	$h_0^5e_4$	$xh_0^5e_4$	$h_0^6e_6$	$xh_0^6e_6$	$h_0^3e_8$	$xh_0^3e_8$	$h_0^4e_{10}$
5		xh_0^5	$h_0^5e_2$	$xh_0^5e_2$	$h_0^4e_4$	$xh_0^4e_4$	$h_0^5e_6$	$xh_0^5e_6$	$h_0^2e_8$	$xh_0^2e_8$	xPh_1 $h_0^3e_{10}$
4		xh_0^4	$h_0^4e_2$	$xh_0^4e_2$	$h_0^3e_4$	$xh_0^3e_4$	$h_0^4e_6$	$xh_0^4e_6$	$xh_0^3h_3$ h_0e_8	xh_0e_8 $h_0^3h_3e_2$	xh_1c_0 $xh_0^3h_3e_2$ $h_0^2e_{10}$
3		xh_0^3	$h_0^3e_2$	$xh_0^3e_2$	$xh_0^2h_2$ $h_0^2e_4$	$xh_0^2e_4$	$h_0^3e_6$	$xh_0^3e_6$	$xh_0^2h_3$ e_8	xc_0 xe_8 $h_0^2h_3e_2$	xh_2^3 $xh_0^2h_3e_2$ h_0e_{10}
2		xh_0^2	$h_0^2e_2$	xh_1^2 $xh_0^2e_2$	xh_0h_2 h_0e_4	xh_0e_4 $h_0h_2e_2$	$xh_0h_2e_2$ $h_0^2e_6$	xh_2^2 $xh_0^2e_6$	xh_0h_3 $h_1^2e_6$	xh_1h_3 $xh_1^2e_6$ $h_0h_3e_2$	$xh_0h_3e_2$ e_{10}
1		xh_0	xh_1 h_0e_2	xh_0e_2	xh_2 e_4	xe_4 h_2e_2	xh_2e_2 h_0e_6	xh_0e_6 h_1e_6	xh_3 xh_1e_6	h_3e_2	xh_3e_2
0		x	e_2	xe_2			e_6	xe_6			
	0	1	2	3	4	5	6	7	8	9	10
											$t-s$

Recall a well known fact (*cf.* [4, Lemma 3.1.3]):

2.3.6 *If $\alpha \in \pi_*^S(BC_{2^n})$ is detected by an element a in $E_2^{*,*}(BC_{2^n})$, then 2α is detected by ah_0.*

Since BC_{2^n} is a Hopf space (*cf.* [2]), the following holds (*cf.* [4, Theorem 2.3.3]).

2.3.7 *The differentials of the* mod 2 *Adams spectral sequence for BC_{2^n} are derivations.*

By (2.1.1) and Theorem 2.2.2, the TR-groups in Theorem 2.1.3 give rise to the stable homotopy groups $\pi_k^S(BC_{2^n})$ for $k \leq 5$ as follows:

$$\begin{array}{rcl} \pi_1^S(BC_{2^n}) & \cong & \mathbb{Z}/2^n, \\ \pi_2^S(BC_{2^n}) & \cong & \mathbb{Z}/2, \\ \pi_3^S(BC_{2^n}) & \cong & \mathbb{Z}/2 \oplus \mathbb{Z}/2^{n+1}, \\ \pi_4^S(BC_{2^n}) & \cong & \mathbb{Z}/2^{\min\{3,n\}}, \\ \pi_5^S(BC_{2^n}) & \cong & \mathbb{Z}/2^{\oplus \min\{1,n-2\}} \oplus \mathbb{Z}/2^n. \end{array} \tag{2.3.8}$$

We obtain the following lemma from (2.3.8).

Lemma 2.3.9. *In the* mod 2 *Adams spectral sequence for BC_{2^n}, the elements x, xe_2 and xe_4 are permanent cycles,*

$$d_n(e_2) = xh_0^n,\ d_n(e_4) = xh_0^{n+1}e_2 \text{ and } d_2(e_6) = \begin{cases} h_0h_2e_2 + xh_0e_4 & n = 2, \\ h_0h_2e_2 & n > 2. \end{cases}$$

Furthermore, $d_2(h_2e_2) = xh_0^2h_2$ if $n = 2$, and h_2e_2 is a permanent cycle otherwise.

Lemma 2.3.10. *The elements h_1e_6 and xh_0e_6 of $E_2^{1,8}(BC_{2^n})$ are permanent cycles.*

Proof. We note that $\pi_6^S(BC_{2^n}) \cong \mathbb{Z}/2$ by Corollary 2.2.10. Since xh_2e_2 is a permanent cycle by Lemma 2.3.9, it detects a generator of $\pi_6^S(BC_{2^n})$, and so h_0e_6 supports a nonzero differential. We deduce $d_n(h_0e_6) = xh_0^ne_4$ from the structure of $\pi_5^S(BC_{2^n})$ in (2.3.8). Therefore $h_0^ie_6$ for $i \geq 1$ cannot be a target of any differential. □

Lemma 2.3.11. $d_n(e_8) = xh_0^{n+3}e_6$.

Proof. By (2.3.4), (2.3.7), Lemmas 2.3.9 and 2.3.10, the elements xh_0e_6, h_1e_6 and xh_2^2 (resp. xh_1e_6, xh_3 and $h_1^2e_6$) detect generators of $\pi_7^S(BC_{2^n})$ (resp. $\pi_8^S(BC_{2^n})$). Since $|\pi_7^S(BC_{2^n})| = 2^{n+4}$ by Corollary 2.2.10, and the elements h_1e_6 and xh_2^2 generate the $\mathbb{Z}/2$-summands, the element detected by $xh_0^{n+3}e_6$ is zero in the homotopy. □

Proposition 2.3.12. $\pi_7^S(BC_{2^n}) \cong \mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/2^{n+2}$. *The generators of summands are detected by xh_2^2, h_1e_6 and xh_0e_6, respectively.*

Lemma 2.3.13. *The element $h_3e_2 \in E_2^{1,10}(BC_{2^n})$ is a permanent cycle if $n > 3$, and $d_n(h_3e_2) = xh_0^nh_3$ if $n = 2, 3$. The element xe_8 is a permanent cycle.*

Proof. Since $d_n(e_2) = xh_0^n$ by Lemma 2.3.9, we have $d_n(h_3e_2) = xh_0^nh_3$ by (2.3.7), which is not zero if $n = 2, 3$, and zero if $n > 3$. By (2.3.7) and Lemma 2.3.11, $h_0^ie_8$ supports a nontrivial differential, and so it cannot be a target of an Adams differential. Therefore $d_r(h_3e_2) = 0$ for $r > n$ in the case for $n > 3$.

Since $d_n(xe_8) = 0$ by Lemma 2.3.11, we see that $d_r(xe_8) = 0$ for $r > n$ similarly. □

This together with Lemma 2.3.10 implies the following result.

Proposition 2.3.14. *$\pi_8^S(BC_{2^n}) \cong \mathbb{Z}/2^{\oplus 2} \oplus \mathbb{Z}/2^{\min\{n,4\}}$. The generators of summands are detected by $xh_1e_6, h_1^2e_6$ and xh_3, respectively.*

Lemma 2.3.15. $|\pi_9^S(BC_{2^n})| = 2^{\min\{2n+2,n+6\}}$.

Proof. Proposition 2.3.14 shows that $|\pi_8^S(BC_{2^n})| = 2^{\min\{n+2,6\}}$, which implies that the undetermined differentials in Lemma 2.2.9 turn out to be trivial. We now see the lemma by the same argument as the proof of Corollary 2.2.10. □

Proposition 2.3.16. *$\pi_9^S(BC_{2^n}) \cong \mathbb{Z}/2^{\oplus 3} \oplus \mathbb{Z}/2^{\min\{n,4\}} \oplus \mathbb{Z}/2^{n-1}$. The generators of summands are detected by $xc_0, xh_1^2e_6, xh_1h_3, h_0^{\max\{4-n,0\}}h_3e_2$ and xe_8, respectively.*

Proof. Since $d_2(xh_3e_2) = 0$ by Lemma 2.3.13, we see that xc_0 and xh_1h_3 generate $\mathbb{Z}/2$-summands by (2.3.4) and (2.3.7). The element $xh_1^2e_6$ detects a generator of the other $\mathbb{Z}/2$ summand by Lemma 2.3.10. Lemma 2.3.13 shows that $h_0^{\max\{4-n,0\}}h_3e_2$ generates the summand $\mathbb{Z}/2^{\min\{n,4\}}$. Lemmas 2.3.13 and 2.3.15 imply that xe_8 generates the summand $\mathbb{Z}/2^{n-1}$. □

Remark 2.3.17. This also implies a differential $d_n(e_{10}) = xh_0^{n-1}e_8$ for $n > 2$, and $d_2(e_{10}) \equiv xh_0e_8 \bmod (h_0^3h_3e_2)$ for $n = 2$.

Bibliography

[1] L. Hesselholt, On the Whitehead spectrum of the circle, Algebraic topology, Abel Symp. Vol. 4, 2009, pp. 131–184.

[2] K. Ishiguro, Classifying spaces and homotopy sets of axes of parings, Proc. Amer. Math. Soc. **124** (1996), 3897–3903.

[3] A. Liulevicius, A theorem in homological algebra and stable homotopy of projective spaces, Trans. Amer. Math. Soc. **109** (1963), 540–552.

[4] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.

[5] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N. J. 1962.

Chapter 3

The first line of the Bockstein spectral sequence on a monochromatic spectrum at an odd prime

The chromatic spectral sequence is introduced in [8] to compute the E_2-term of the Adams-Novikov spectral sequence for computing the stable homotopy groups of spheres. The E_1-term $E_1^{s,t}(k)$ of the spectral sequence is an Ext group of BP_*BP-comodules. There are a sequence of Ext groups $E_1^{s,t}(n-s)$ for non-negative integers n with $E_1^{s,t}(0) = E_1^{s,t}$, and Bockstein spectral sequences computing a module $E_1^{s,*}(n-s)$ from $E_1^{s-1,*}(n-s+1)$. So far, a small number of the E_1-terms are determined. Here, we determine the $E_1^{1,1}(n-1) = \mathrm{Ext}^1 M_{n-1}^1$ for $p > 2$ and $n > 3$ by computing the Bockstein spectral sequence with E_1-term $E_1^{0,s}(n)$ for $s = 1, 2$. As an application, we study the non-triviality of the action of α_1 and β_1 in the homotopy groups of the second Smith-Toda spectrum $V(2)$. This is a joint work with Professor Shimomura.

3.1 Introduction

Let p be a prime number, $\mathcal{S}_p$ the stable homotopy category of p-local spectra, and S the sphere spectrum localized at p. Understanding homotopy groups $\pi_*(S)$ of S is one of the principal problems in stable homotopy theory. The main vehicle for computing $\pi_*(S)$ is the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum BP. BP is the p-typical component of MU, the complex cobordism spectrum, and that it has homotopy groups $BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ where v_n is a canonical generator of degree $2p^n - 2$. In order to study the E_2-term of the Adams-Novikov spectral sequence, H. Miller, D. Ravenel and S. Wilson [8] introduced the chromatic spectral sequence. It was designed to compute the E_2-term, but has the following deeper connotation. Let $L_n\colon \mathcal{S}_p \to \mathcal{S}_p$ denote the Bousfield-Ravenel localization functor with respect to $v_n^{-1}BP$ (*cf.* [12]). It gives rise the chromatic filtration $\mathcal{S}_p \to \cdots \to L_n\mathcal{S}_p \to L_{n-1}\mathcal{S}_p \to \cdots \to L_0\mathcal{S}_p$ of the stable homotopy category of spectra, which is a powerful tool for understanding the category. The chromatic nth layer of the spectrum S can be determined from the homotopy groups of $L_{K(n)}S$, the Bousfield localization of S with respect to the nth Morava K-theory $K(n)$ that it has homotopy groups $K(n)_* = v_n^{-1}\mathbb{Z}/p[v_n]$ for $n > 0$ and $K(0)_* = \mathbb{Q}$. By the chromatic convergence theorem of Hopkins-Ravenel [13], S is the inverse limit of the L_nS. Let $E(n)$ be the nth Johnson-Wilson spectrum $E(n)$ with $E(n)_* = v_n^{-1}\mathbb{Z}_{(p)}[v_1, \cdots, v_n]$ for $n > 0$ and $E(0) = K(0)$. It is Boufield equivalent to $v_n^{-1}BP$ and also to $K(0) \vee \cdots \vee K(n)$, i.e. $L_{E(n)} = L_n = L_{K(0)\vee\cdots\vee K(n)}$. We notice that $E(0) = H\mathbb{Q}$, the rational Eilenberg-MacLane spectrum, and $E(1)$ is the p-local Adams summand of periodic complex K-theory. Futhermore, $E(2)$ is closely related to elliptic cohomology. So far, we have no geometric interpretation of homology theories $K(n)$ or $E(n)$ when $n > 2$.

From now on, we assume that the prime p is odd. We explain the E_1-term of the chromatic spectral sequence. The Brown-Peterson spectrum BP is a ring spectrum that induces the Hopf algebroid $(BP_*, BP_*(BP)) = (BP_*, BP_*[t_1, t_2, \dots])$ in the standard way [14], and we have an induced Hopf algebroid

$$(E(n)_*, E(n)_*(E(n))) = (E(n)_*, E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*)$$

where $E(n)_*$ is considered to be a BP_*-module by sending v_k to zero for $k > n$. Then, the E_1-term is given by

$$E_1^{s,t}(n-s) = \mathrm{Ext}^t_{E(n)_*(E(n))}(E(n)_*, M^s_{n-s}).$$

Here, M^s_{n-s} denotes the $E(n)_*(E(n))$-comodule $E(n)_*/(I_{n-s}+(v^\infty_{n-s}, v^\infty_{n-s+1}, \dots, v^\infty_{n-1}))$, in which I_k denotes the ideal of $E(n)_*$ generated by v_i for $0 \le i < k$ $(v_0 = p)$, and $M/(w^\infty)$ for $w \in E(n)_*$ and an $E(n)_*$-module M denotes the cokernel of the localization map $M \to w^{-1}M$. In order to study the stable homotopy groups $\pi_*(L_{K(n)}S)$, we study here the homotopy groups of the monochromatic component M_nS of S (see [12]). Then, the E_2-term $E_2^{s,t}(M_nS)$ of the Adams-Novikov spectral sequence for computing $\pi_*(M_nS)$ is the E_1-term $E_1^{n,s}(0)$ of the chromatic spectral sequence. In [8], the authors also introduced the v_{n-s}-Bockstein spectral sequence $E_1^{s-1,t+1}(n-s+1) \Rightarrow E_1^{s,t}(n-s)$ associated to a short exact sequence

$$0 \to M^{s-1}_{n-s+1} \xrightarrow{\varphi} M^s_{n-s} \xrightarrow{v_{n-s}} M^s_{n-s} \to 0$$

of $E(n)_*(E(n))$-comodules, where $\varphi(x) = x/v_{n-s}$. So far, the E_1-term $E_1^{s,t}(n-$

s) is determined in the following cases (*cf.* [14]):

$$
\begin{aligned}
(s,t,n) &= (0,t,n) \quad \text{for (a) } n \le 2, \text{ (b) } n=3,\ p>3, \text{ (c) } t \le 2 \text{ by Ravenel [11],} \\
&\qquad\quad \text{(Henn [2] for } n=2 \text{ and } p=3\text{),} \\
&= (1,0,n) \quad \text{for } n \ge 0 \text{ by Miller, Ravenel and Wilson [8],} \\
&= (s,t,n) \quad \text{for } n \le 2 \text{ by Shimomura and his colaborators: Arita [1],} \\
&\qquad\quad \text{Tamura [20], Yabe [21] and Wang [22], ([15], [18], [19]),} \\
&= (1,1,3) \quad \text{by Shimomura [16], Hirata and Shimomura [3],} \\
&= (2,0,n) \quad \text{for } n>3 \text{ by Shimomura [17], for } n=3 \text{ by Nakai [9], [10].}
\end{aligned}
$$

In this chapter, we determine the structure of $E_1^{1,1}(n-1)$ for $n>3$. The case $n=3$, which is special, is treated in [16] and [3]. The result is the first step to understand $\pi_*(L_{K(n)}S)$ for $n>3$ as explained above. We proceed to state the result.

In this chapter, we consider only the cases $s=0$ and $s=1$, and, hereafter, put

$$v = v_n \quad \text{and} \quad u = v_{n-1}.$$

Furthermore, we put

$$F = \mathbb{Z}/p,$$

and consider the coefficient ring $K(n)_* = F[v_n^{\pm 1}] = F[v^{\pm 1}] = E(n)_*/I_n$,

$$A = E(n)_*/I_{n-1} \quad \text{and} \quad B = M_{n-1}^1 = A/(u^\infty) = \text{Coker } (A \to u^{-1}A).$$

Since the ideal I_{n-1} is invariant, $(A,\Gamma) = (A, E(n)_*(E(n))/I_{n-1})$ is a Hopf algebroid, and we use the abbreviation

$$\text{Ext}^s M = \text{Ext}^s_\Gamma(A, M)$$

for a Γ-comodule M. Then, the chromatic E_1-terms are

$$E_1^{0,t}(n) = \text{Ext}^t K(n)_* \quad \text{and} \quad E_1^{1,t}(n-1) = \text{Ext}^t B.$$

We have the u-Bockstein spectral sequence

(3.1.1) $$E_1 = \text{Ext}^* K(n)_* \Longrightarrow \text{Ext}^* B$$

associated to the short exact sequence

(3.1.2) $$0 \xrightarrow{K} (n)_* \xrightarrow{\varphi} B \xrightarrow{u} B \to 0,$$

where φ is a homomorphism defined by $\varphi(x) = x/u$.

Let R be a ring, and let $R\langle g\rangle$ denote the R-module generated by g. The E_1-term of the u-Bockstein spectral sequence was determined by Ravenel [11] as follows:

Theorem 3.1.3. $\text{Ext}^0 K(n)_* = K(n)_*$ *and*

$$
\begin{aligned}
\text{Ext}^1 K(n)_* &= K(n)_*\langle h_i, \zeta_n : 0 \le i < n\rangle, \\
\text{Ext}^2 K(n)_* &= K(n)_*\langle \zeta_n h_i, b_i, g_i, k_i, h_j h_k : 0 \le i < n,\ 0 \le j < k-1 < n-1\rangle.
\end{aligned}
$$

In the theorem, the generators h_i and b_i are represented by $t_1^{p^i}$ and $\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}t_1^{kp^i}\otimes t_1^{(p-k)p^i}$ of the cobar complex $\Omega_\Gamma^* K(n)_*$, respectively, and g_i and k_i are given by the Massey products

$$g_i = \langle h_i, h_i, h_{i+1}\rangle \quad \text{and} \quad k_i = \langle h_i, h_{i+1}, h_{i+1}\rangle. \tag{3.1.4}$$

In order to determine the module $\mathrm{Ext}^0 B$, Miller, Ravenel and Wilson [8] introduced elements x_i and integers a_i in [8, (5.11) and (5.13)], where they denoted them by $x_{n,i}$ and $a_{n,i}$, such that $x_i \equiv v^{p^i} \mod I_n$ with the action of the connecting homomorphism δ given in [8, (5.18)]:

$$\delta(v^s/u) = sv^{s-1}h_{n-1} \quad \text{and} \quad \delta(x_i^s/u^{a_i}) = sv^{(sp-1)p^{i-1}}h_{[i-1]} \text{ for } i \geq 1. \tag{3.1.5}$$

Hereafter, we let

$$[i] \in \{0, 1, \dots, n-2\}$$

be the principal representative of the integer i module $n-1$. The elements x_i and the integers a_i are defined inductively by $x_0 = v$ and $a_0 = 1$, and for $i > 0$,

$$\begin{aligned} x_i &= \begin{cases} x_{i-1}^p & \text{for } i = 1 \text{ or } [i] \neq 1, \\ x_{i-1}^p - u^{b_{n,i}}v^{p^i - p^{i-1}+1} & \text{for } i > 1 \text{ and } [i] = 1, \text{ and} \end{cases} \\ a_i &= \begin{cases} pa_{i-1} & \text{for } i = 1 \text{ or } [i] \neq 1, \\ pa_{i-1} + p - 1 & \text{for } i > 1 \text{ and } [i] = 1. \end{cases} \end{aligned} \tag{3.1.6}$$

Here, $b_{n,k(n-1)+1} = (p^n-1)(p^{k(n-1)}-1)/(p^{n-1}-1)$. The result (3.1.5) determines the differentials of the Bockstein spectral sequence, which implies:

Theorem 3.1.7. ([8, Th. 5.10]) *As a k_*-module,*

$$\mathrm{Ext}^0 B = L_\infty \oplus \bigoplus_{p\nmid s, i\geq 0} L_{a_i}\langle x_i^s\rangle.$$

Here, $k_* = k(n-1)_* = F[u]$, $L_i = k_*/(u^i)$ and $L_\infty = k_*/(u^\infty) = \mathrm{colim}_i L_i$.

This theorem together with (3.1.5) implies the following:

Corollary 3.1.8. *The cokernel of $\delta\colon \mathrm{Ext}^0 B \to \mathrm{Ext}^1 K(n)_*$ is the F-module generated by*

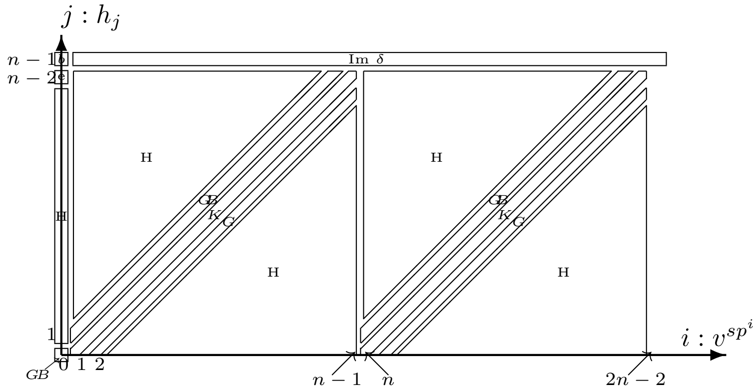
$$\begin{array}{ll} v^t\zeta_n, & v^{tp-1}h_{n-1}, \quad h_j \quad \textit{for } 0 \leq j < n-1, \textit{ and} \\ v^{sp^k}h_j & \textit{for } 0 \leq j < n-1, \textit{ where } [k] \neq [j],\ s \not\equiv -1\ (p),\ \textit{or } s \equiv -1\ (p^2), \end{array}$$

for integers s and t with $p \nmid s$.

By Theorem 3.1.3, the module $\mathrm{Ext}^1 K(n)_*$ is the direct sum of $\zeta_n \mathrm{Ext}^0 K(n)_* = \zeta_n K(n)_*$, $F\langle h_j\rangle$ for $j \in \mathbb{Z}/(n-1)$ and the modules

$$V_{(i,j,s)} = F\langle v^{sp^i}h_j\rangle$$

for $(i,j,s) \in \mathbb{N} \times \mathbb{Z}/n \times \overline{\mathbb{Z}}$. Here, $\mathbb{N}$ denotes the set of non-negative integers, and $\overline{\mathbb{Z}} = \mathbb{Z} \setminus p\mathbb{Z}$. We partition $\mathbb{N} \times \mathbb{Z}/n$ as follows:



More precisely,

$$
\begin{array}{rcl}
H & = & \{(0,j) : 1 \le j < n-2\} \\
 & & \cup\{(i,j) : i > 0,\ [i] \neq n-3, n-2,\ 2+[i] \le j \le n-2\} \\
 & & \cup\{(i,j) : i > 0,\ [i] \neq 0, 1,\ 0 \le j \le [i]-2\}, \\
GB & = & \{(i,[i]) : i \ge 0\}, \\
K & = & \{(i,[i]-1) : i > 0,\ [i] \neq 0\} \quad \text{and} \\
G & = & \{(i,[i]-2) : i > 1,\ [i] \neq 0, 1\}.
\end{array}
$$

We introduce notation

$$
\begin{array}{rcl}
V_{(0,n-2)} & = & \bigoplus_{s \in \overline{\mathbb{Z}}'} V_{(0,n-2,s)}, \\
V_{(0,n-1)} & = & \bigoplus_{t \in \mathbb{Z}} V_{(0,n-1,tp-1)} \ = \ F[v^{\pm p}]\langle v^{-1} h_{n-1} \rangle, \\
C_X & = & \bigoplus_{(i,j) \in X,\ s \in \overline{\mathbb{Z}}} V_{(i,j,s)} \quad \text{for a subset } X \subset \mathbb{N} \times \mathbb{Z}/n, \\
\overline{C}_{GB} & = & \bigoplus_{(i,j) \in GB} \left(\left(\bigoplus_{s \in \overline{\overline{\mathbb{Z}}}} V_{(i,j,s)} \right) \oplus \left(\bigoplus_{t \in \mathbb{Z}} V_{(i,j,tp^2-1)} \right) \right) \\
 & = & \bigoplus_{(i,[i],s) \in \widetilde{GB}} V_{(i,j,s)} \oplus \bigoplus_{i \ge 0} F[v^{\pm p^{i+2}}]\langle v^{-p^i} h_{[i]} \rangle \quad \text{and} \\
C_O & = & F\langle \theta, h_j : j \in \mathbb{Z}/(n-1) \rangle.
\end{array}
$$

Here, for $e(i) = (p^i - 1)/(p-1)$, $\theta = v^{e(n-2)} h_{n-2}$,

$$
\begin{array}{rcl}
\overline{\mathbb{Z}}' & = & \overline{\mathbb{Z}} \setminus \{e(n-2)\}, \quad \overline{\overline{\mathbb{Z}}} \ = \ \{n \in \overline{\mathbb{Z}} : p \nmid (s+1)\} \quad \text{and} \\
\widetilde{GB} & = & \{(i,[i],s) : s \in \overline{\overline{\mathbb{Z}}}\}.
\end{array}
$$

We also consider the subset $\boldsymbol{T}$ of $\mathbb{N} \times \mathbb{Z}/n \times \overline{\mathbb{Z}}$ defined by

$$
\begin{array}{rcl}
\boldsymbol{T} & = & \{(i,j,s) \in \mathbb{N} \times \mathbb{Z}/n \times \overline{\mathbb{Z}} :\ p \nmid (s+1) \text{ or } p^2 \mid (s+1) \text{ if } [i] = j, \\
 & & \quad p \mid (s+1) \text{ if } (i,j) = (0,n-1), \text{ and } s \neq e(n-2) \text{ if } (i,j) = (0,n-2)\}.
\end{array}
$$

In this notation, the cokernel of δ in Corollary 3.1.8 is given by
(3.1.9)

$$
\begin{array}{rcl}
\text{Coker } \delta & = & \zeta_n K(n)_* \oplus C_O \oplus \bigoplus_{(i,j,s) \in \boldsymbol{T}} V_{(i,j,s)} \\
 & = & \zeta_n K(n)_* \oplus C_O \oplus V_{(0,n-2)} \oplus V_{(0,n-1)} \oplus C_H \oplus C_K \oplus C_G \oplus \overline{C}_{GB}
\end{array}
$$

Finally, we consider the k_*-modules:

$$\begin{array}{rcl} W_{(i,j,s)} & = & L_{a(i,j,s)}\langle x_i^s h_j\rangle, \\ W_{(0,n-2)} & = & \bigoplus_{s\in\overline{\mathbb{Z}}'} W_{(0,n-2,s)}, \\ W_{(0,n-1)} & = & \bigoplus_{t\in\mathbb{Z}} W_{(0,n-1,tp-1)}, \\ B_X & = & \bigoplus_{(i,j)\in X,\ s\in\overline{\mathbb{Z}}} W_{(i,j,s)} \quad \text{for a subset } X\subset \mathbb{N}\times\mathbb{Z}/n, \\ \overline{B}_{GB} & = & \bigoplus_{(i,j)\in GB}\Big(\Big(\bigoplus_{s\in\overline{\overline{\mathbb{Z}}}} W_{(i,j,s)}\Big)\oplus\big(\bigoplus_{t\in\mathbb{Z}} W_{(i,j,tp^2-1)}\big)\Big) \quad \text{and} \\ C_\infty & = & (K(n-1)_*/k_*)\,\langle\theta, h_j : j\in\mathbb{Z}/(n-1)\rangle. \end{array}$$

Here, $a(i,j,s)$ denotes an integer defined as follows: for $(i,j)=(0,n-2)$, $a(0,n-2,s)=2$ if $p\nmid s(s-1)$, and

$$a(0,n-2,s) \;=\; \begin{cases} a_l & p\nmid t,\ l>0,\ [l]\neq 0,n-2, \\ a_l+e(n-2)+p^{n-3} & p\nmid t,\ l>0,\ [l]=n-2, \\ a_l+1 & p\nmid t,\ l>0,\ [l]=0 \end{cases}$$

if $s=tp^l+e(n-2)$; for $(i,j)\in\{(0,n-1)\}\cup H\cup K\cup G\cup GB$,

$$a(i,j,s) \;=\; \begin{cases} p-1 & (i,j)=(0,n-1), \\ a_i & (i,j)\in H, \\ a_i+a_{i-1} & (i,j)\in K\cup G, \\ 2a_i & (i,j,s)\in\widetilde{GB}, \\ (p-1)a_{i+1} & (i,j)\in GB,\ p^2\mid(s+1). \end{cases}$$

Theorem 3.1.10. *The chromatic E_1-term $\mathrm{Ext}^1 B=\mathrm{Ext}^1 M^1_{n-1}$ is canonically isomorphic to the k_*-module*

$$\zeta_n\mathrm{Ext}^0 B\oplus C_\infty\oplus W_{(0,n-2)}\oplus W_{(0,n-1)}\oplus B_H\oplus B_K\oplus B_G\oplus\overline{B}_{GB}.$$

Let $V(n)$ be the nth Smith-Toda spectrum defined by $BP_*(V(n))=BP_*/I_{n+1}$. As an application of the theorem, we study the action of α_1 and β_1 on the elements u^t $(t>0)$ in the Adams-Novikov E_2-term $E_2^*(V(n))$ in section 6. In particular, it leads us an geometric result for $n=4$. In [23], Toda constructed the self map γ on $V(2)$ to show the existence of $V(3)$ for the prime $p>5$. We notice that $\gamma^t i\in\pi_*(V(2))$ for the inclusion $i\colon S\to V(2)$ to the bottom cell is detected by $u^t=v_3^t\in BP_*(V(2))$ in the Adams-Novikov spectral sequence.

Theorem 3.1.11. *Let $p>5$. Then $\gamma^t i\alpha_1$ and $\gamma^t i\beta_1$ are nontrivial in $\pi_*(V(2))$ for $t>0$.*

3.2 Bockstein spectral sequence

We compute the Bockstein spectral sequence by use of the following lemma.

Lemma 3.2.1. *Let $\delta\colon \mathrm{Ext}^s B \to \mathrm{Ext}^{s+1} K(n)_*$ be the connecting homomorphism associated to the short exact sequence (3.1.2). Suppose that $\mathrm{Coker}\ \delta = \bigoplus_k V_k \subset \mathrm{Ext}^1 K(n)_*$ and $\bigoplus_k U_k \subset \mathrm{Ext}^2 K(n)_*$ for F-modules V_k and U_k, and there exist u-torsion k_*-modules W_k fitting in a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & V_k \arrow[r, "\varphi'_*"] \arrow[d] & W_k \arrow[r, "u"] \arrow[d, "f_k"'] & W_k \arrow[r, "\delta'"] \arrow[d, "f_k"] & U_k \arrow[d] \\
0 \arrow[r] & \mathrm{Coker}\ \delta \arrow[r, "\varphi_*"] & \mathrm{Ext}^1 B \arrow[r, "u"] & \mathrm{Ext}^1 B \arrow[r, "\delta"] & \mathrm{Ext}^2 K(n)_*
\end{tikzcd}$$

of exact sequences. Then, $\mathrm{Ext}^1 B = \bigoplus_k W_k$.

This follows immediately from [8, Remark 3.11].

Let $\widetilde{\theta}$ be an element of Corollary 3.5.8. Then, $\widetilde{\theta}/u^k$ and h_j/u^k for $j \in \mathbb{Z}/(n-1)$ belong to $\mathrm{Ext}^1 B$, and we define the map $f\colon C_\infty \to \mathrm{Ext}^1 B$ by $f((u^{-k})\theta) = \widetilde{\theta}/u^k$ and $f((u^{-k})h_j) = h_j/u^k$ for $(u^{-k}) \in K(n-1)_*/k_*$, so that the short exact sequence

$$0 \to C_O \xrightarrow{1/u} C_\infty \xrightarrow{u} C_\infty \to 0 \tag{3.2.2}$$

yields a summand of Lemma 3.2.1.

Note that if a cocycle z represents ζ_n, then so does z^p. Therefore, we have $\zeta_n/u^j \in \mathrm{Ext}^1 B$ represented by z^{p^j}/u^j. The exact sequence (3.1.2) induces the exact sequence $0 \to \mathrm{Ext}^0 K(n)_* \xrightarrow{\varphi_*} \mathrm{Ext}^0 B \xrightarrow{u} \mathrm{Ext}^0 B \xrightarrow{\delta} \mathrm{Ext}^1 K(n)_*$, and we have an exact sequence

$$0 \to \zeta_n \mathrm{Ext}^0 K(n)_* \xrightarrow{\varphi_*} \zeta_n \mathrm{Ext}^0 B \xrightarrow{u} \zeta_n \mathrm{Ext}^0 B \xrightarrow{\delta} \zeta_n \mathrm{Ext}^1 K(n)_*, \tag{3.2.3}$$

which is a summand of Lemma 3.2.1. Together with (3.2.2) and (3.2.3), Theorem 3.1.10 follows from Lemma 3.2.1 if the following sequence is exact for each $(i,j,s) \in \boldsymbol{T}$:

$$0 \to V_{(i,j,s)} \xrightarrow{\varphi'_*} W_{(i,j,s)} \xrightarrow{u} W_{(i,j,s)} \xrightarrow{\delta'} U_{(i,j,s)}, \tag{3.2.4}$$

where $U_{(i,j,s)}$ denotes an F-module generated by a single generator as follows: for $(i,j) = (0, n-2)$, $U_{(0,n-2,s)} = \mathbb{F}_p v^{s-2} k_{n-2}$ if $p \nmid s(s-1)$,

$$U_{(0,n-2,s)} = \begin{cases} \mathbb{F}_p v^{s-p^{l-1}} h_{[l-1]} h_{n-2} & p \nmid t,\ l > 0,\ [l] \neq 0, n-2, \\ \mathbb{F}_p v^{s-p^{l-1}} b_{2n-5} & p \nmid t,\ l > 0,\ [l] = n-2, \\ \mathbb{F}_p v^{s-p^{l-1}-1} g_{n-2} & p \nmid t,\ l > 0,\ [l] = 0; \end{cases}$$

if $s = tp^l + e(n-2)$; for $(i,j) \in \{(0, n-1)\} \cup H \cup K \cup G \cup GB$,

$$U_{(i,j,s)} = \begin{cases} \mathbb{F}_p v^{s-p+1} b_{n-1} & (i,j) = (0, n-1), \\ F\langle v^{(sp-1)p^{i-1}} h_{[i-1]} h_j \rangle & (i,j) \in H, \\ \mathbb{F}_p v^{(s-2)p} k_{n-1} & (i,j) = (1,0) \in K, \\ \mathbb{F}_p v^{(sp^2-p-1)p^{i-2}} k_{[i-2]} & (i,j) \in K,\ i > 1, \\ \mathbb{F}_p v^{(sp^2-p-1)p^{i-2}} g_{[i-2]} & (i,j) \in G, \\ \mathbb{F}_p v^{s-p-1} g_{n-1} & (i,j,s) \in \widetilde{GB},\ i = 0, \\ \mathbb{F}_p v^{(sp-2)p^{i-1}} g_{[i-1]} & (i,j,s) \in \widetilde{GB},\ i > 0, \\ F\langle v^{(s+1-p)p^i} b_j \rangle & (i,j) \in GB,\ p^2 \mid (s+1). \end{cases}$$

Since the mapping $\boldsymbol{T} \to \{U_{(i,j,s)} : (i,j,s) \in \boldsymbol{T}\}$ assigning (i,j,s) to $U_{(i,j,s)}$ is an injection, we see the following:

Lemma 3.2.5. *The direct sum of $\zeta_n \mathrm{Ext}^1 K(n)_*$ and $U_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is a sub-F-module of $\mathrm{Ext}^2 K(n)_*$.*

The homomorphism f_k in Lemma 3.2.1 on $W_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is explicitly given by

$$f_{(i,j,s)}(x) = x/u^{a(i,j,s)}.$$

It follows that the homomorphism δ' on it is given by the composite $\delta(1/u^{a(i,j,s)})$. Hereafter we denote it by $\delta'_{(i,j,s)}$, that is, $\delta'_{(i,j,s)} = \delta(1/u^{a(i,j,s)})$, and consider a condition:

$(3.2.6)_{(i,j,s)}$ $\quad \delta'_{(i,j,s)}(x) = y$ for the generators $x \in W_{(i,j,s)}$ and $y \in U_{(i,j,s)}$.

Note that $\varphi'_*(\overline{x}) = u^{a(i,j,s)-1} x$ for the generators $\overline{x} \in V_{(i,j,s)}$ and $x \in W_{(i,j,s)}$, since $f_k \varphi'_*(\overline{x}) = \varphi_*(\overline{x}) = x/u$. Then,

Lemma 3.2.7. *For each $(i,j,s) \in \boldsymbol{T}$, if the condition $(3.2.6)_{(i,j,s)}$ holds, then (3.2.4) for (i,j,s) is exact and yields a summand of Lemma 3.2.1.*

The relations in (3.1.5) show immediately

(3.2.8) *The condition $(3.2.6)_{(i,j,s)}$ holds for $(i,j) \in H$.*

Proof of Theorem 3.1.10. The theorem follows from Lemmas 3.2.1, 3.2.5 and 3.2.7 together with (3.2.2), (3.2.3), (3.2.8), Lemmas 3.3.7, 3.3.8, 3.4.1 and 3.5.9, in which the lemmas are proved below. Indeed, the direct sum of $\zeta_n \mathrm{Ext}^0 K(n)_*$, C_O and $V_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is the cokernel of δ by (3.1.9). $\square$

3.3 The summands on $V_{(0,n-1)}$ and $\overline{C}_{GB}$

We begin with stating some formulae on the Hopf algebroid (A,Γ):
(3.3.1)

$$\begin{aligned} 0 &= vt_k^{p^n} + ut_{k+1}^{p^{n-1}} - u^{p^{k+1}}t_{k+1} - t_k\eta_R(v^{p^k}) \in \Gamma \quad \text{for } k<n, \\ \eta_R(u) &= u, \quad \eta_R(v) \;=\; v + ut_1^{p^{n-1}} - u^p t_1, \\ \Delta(t_k) &= \textstyle\sum_{i=0}^k t_i \otimes t_{k-i}^{p^i} \quad \text{for } k<n, \text{ and} \\ \Delta(t_n) &= \textstyle\sum_{i=0}^n t_i \otimes t_{n-i}^{p^i} - ub_{n-2}. \end{aligned}$$

Then the connecting homomorphism $\delta\colon \mathrm{Ext}^1 B \to \mathrm{Ext}^2 K(n)_*$ is computed by the differential $d\colon \Omega^1_\Gamma A \to \Omega^2_\Gamma A$ of the cobar complex modulo an ideal, which is defined by

$$d(x) = 1\otimes x - \Delta(x) + x\otimes 1. \tag{3.3.2}$$

We also use the differential $d\colon \Omega^0_\Gamma A \to \Omega^1_\Gamma A$ defined by $d(w) = \eta_R(w) - \eta_L(w)$. For $w, w' \in \Omega^0_\Gamma A$ and $x \in \Omega^1_\Gamma A$, these differentials satisfy
(3.3.3)

$$\begin{aligned} d(ww') &= d(w)\eta_R(w') + wd(w'), \; d(wx) \;=\; d(w)\otimes x + wd(x), \text{ and} \\ d(x\eta_R(w)) &= d(x)\eta_R(w) - x\otimes d(w). \end{aligned}$$

We also use the Steenrod operations P^0 and βP^0 on $\mathrm{Ext}^*C(j)$ for $j\ge 1$ and Ext^*B (*cf.* [6], [14]). Here, $C(j)$ denotes the comodule $A/(u^j)$, and we notice that $C(1) = K(n)_*$. Let $\widetilde\Omega^s M = \Omega^s_{E(n)_*(E(n))}M$ for an $E(n)_*(E(n))$-comodule M. Given a cocycle $x(j)$ of $\widetilde\Omega^s C(j)$, $\widetilde x(j)$ denotes a cochain of $\widetilde\Omega^s E(n)_*$ such that $\pi_j(\widetilde x(j)) = x(j)$ for the projection $\pi_j\colon \widetilde\Omega^s E(n)_* \to \widetilde\Omega^s C(j)$. Since $x(j)$ is a cocycle, $d(\widetilde x(j)^p) = py_j + \sum_{i=1}^{n-2} v_i^p z_{j,i} + u^{jp}z_{j,n-1}$ for some elements y_j and $z_{j,i} \in \widetilde\Omega^{s+1}E(n)_*$. Under this situation, the Steenrod operations are defined by

$$\begin{aligned} P^0([x(j)]) = [x(j)^p] \quad &\text{and} \quad \beta P^0([x(j)]) = [y_j] \in \mathrm{Ext}^*C(jp), \quad \text{and} \\ P^0([x(j)/u^j]) = [x(j)^p/u^{jp}] \quad &\text{and} \quad \beta P^0([x(j)/u^j]) = [y_j/u^{jp}] \in \mathrm{Ext}^*B. \end{aligned}$$

Here, $[x]$ denotes the homology class represented by a cocycle x. In particular, the operation acts on our elements as follows:

$$\beta P^0(x_i/u^{a_i}) \;=\; \begin{cases} v^{p-1}h_{n-1}/u^{p-1} & i=0, \\ x_{i-1}^{p^2-1}h_{[i-1]}/u^{(p-1)a_i} & i>0, \end{cases} \quad \text{in } \mathrm{Ext}^1 B; \tag{3.3.4}$$

(3.3.5)

$$\begin{aligned} P^0(x_i^s h_k/u^j) &= \begin{cases} x_{i+1}^s h_{k+1}/u^{jp} & k\ne n-2, \\ x_{i+1}^s h_0/u^{jp-p+1} & k = n-2; \end{cases} \quad \text{in } \mathrm{Ext}^1 B; \text{ and} \\ \beta P^0(x_i^s h_k) &= x_{i+1}^s b_k \quad \text{in } \mathrm{Ext}^2 K(n)_*. \end{aligned}$$

The following is a folklore (cf. [14, Corollary A1.5.5]):

$$P^0\delta = \delta P^0 \quad \text{and} \quad \beta P^0 \delta = -\delta\beta P^0 \quad \text{in } \mathrm{Ext}^* K(n)_*. \tag{3.3.6}$$

Lemma 3.3.7. *The condition* $(3.2.6)_{(i,j,s)}$ *holds for each* $(i,j,s) \in \{(0,n-1,tp-1),(i,j,tp^2-1) : t \in \mathbb{Z}, (i,j) \in GB\}$.

Proof. For $k \geq -1$, consider a generator $x(k,t) = x_k^{tp^2-1}h_{[k]}$ for $k \geq 0$ and $x(-1) = x_0^{tp-1}h_{n-1}$, and $\overline{(k,t)}$ denotes a triple $(k,[k],tp^2-1)$ if $k \geq 0$ and $(0,n-1,tp-1)$ if $k=-1$. Then, $(1/u^{a\overline{(k,t)}})(x(k,t)) = x_{k+2}^{t-1}\beta P^0(x_{k+1}/u^{a_{k+1}})$ for $k \geq -1$ by (3.3.4). Now, $\delta'_{\overline{(k,t)}}(x(k,t))$ equals

$$x_{k+2}^{t-1}\delta(\beta P^0(x_{k+1}/u^{a_{k+1}})) \;=\; -x_{k+2}^{t-1}(\beta P^0(x_k^{p-1}h_{\overline{[k]}})) \;=\; -x_{k+1}^{\nu(t)}b_{\overline{[k]}}$$

by (3.3.6), (3.1.5) and (3.3.5). Here, $(\nu(t),\overline{[k]}) = (tp-1,[k])$ if $k \geq 0$ and $= ((t-1)p, n-1)$ if $k=-1$. □

Lemma 3.3.8. *The condition* $(3.2.6)_{(i,[i],s)}$ *holds for* $(i,[i],s) \in \widetilde{GB}$.

Proof. We prove this by induction on i. By (3.3.1) and (3.3.2), we compute mod (u^3)

$$\begin{aligned} d(v^{s+1-p}t_1^{p^n}) &\equiv (s+1)uv^{s-p}t_1^{p^{n-1}} \otimes t_1^{p^n} + \binom{s+1}{2}u^2v^{s-p-1}t_1^{2p^{n-1}} \otimes t_1^{p^n} \\ d((s+1)uv^{s-p}t_2^{p^{n-1}}) &\equiv s(s+1)u^2v^{s-p-1}t_1^{p^{n-1}} \otimes t_2^{p^{n-1}} - (s+1)uv^{s-p}t_1^{p^{n-1}} \otimes t_1^{p^n} \end{aligned}$$

to obtain $\delta(v^sh_0/u^2) = s(s+1)v^{s-p-1}g_{n-1}$ and so

$$\delta'_{(0,0,s)}(v^sh_0) \;=\; s(s+1)v^{s-p-1}g_{n-1}.$$

Apply P^0 to it, and we obtain

$$\begin{aligned} \delta'_{(1,1,s)}(v^{sp}h_1) &= \delta(P^0(v^sh_0/u^2)) \;=\; P^0\delta(v^sh_0/u^2) \;=\; s(s+1)P^0(v^{s-p-1}g_{n-1}) \\ &= s(s+1)v^{sp-p^2-p}g_n \;=\; s(s+1)v^{sp-2}g_0. \end{aligned}$$

Here, we notice that $g_n = v^{p^2+p-2}g_0$ in $\mathrm{Ext}^2K(n)_*$ by (3.3.1). Suppose inductively that $\delta'_{(i,1,s)}(x_i^sh_1) = s(s+1)v^{(sp-2)p^{i-1}}g_0$ for $[i]=1$, which is $(3.2.6)_{(i,1,s)}$. Note that $a_{i+j} = pa_{i+j-1}$ if $0 < j < n-2$, and we see that $P^0\delta'_{(i,j,s)} = \delta'_{(i+1,j+1,s)}P^0$ by (3.3.6). Therefore, $(P^0)^j$ for $j < n-2$ yields the equation for $\delta'_{a(i+j,j+1,s)}(x_{i+j}^sh_{j+1})$. At $i' = i+n-2$, for $t = (i',0,s)$, $\delta'_t(x_{i'}^sh_0) = \delta P^0(x_{i'-1}^sh_{n-2}/u^{a(i'-1,n-2,s)})$ (by (3.3.5)) $= s(s+1)v^{(sp-2)p^{i+n-3}}g_{n-2}$ by (3.3.6) and inductive hypothesis.

Note that $a_{i+n-1} = p^{n-1}a_i + p - 1$. Consider the connecting homomorphism $\delta_j \colon \mathrm{Ext}^1M_{n-1}^1 \to \mathrm{Ext}^2C(j)$ associated to the short exact sequence $0 \to C(j) \xrightarrow{1/u^j} M_{n-1}^1 \xrightarrow{u^j} M_{n-1}^1 \to 0$. Then, $u^{j-1}\delta = \delta_ju^{j-1}$. Besides, $\delta_j(P^0)^k = (P^0)^k\delta$ if $p^k \geq j$. Now in $\mathrm{Ext}^2C(p^2+p-1)$, $u^{p^2+p-2}\delta'_{(i+n-1,1,s)}(x_{i+n-1}^sh_1)$ equals

$$\begin{aligned} u^{p^2+p-2}\delta(x_{i+n-1}^sh_1/u^{p^{n-1}a+2(p-1)}) &= \delta_{p^2+p-1}(P^0)^{n-1}(x_i^sh_1/u^a) \\ = (P^0)^{n-1}(s(s+1)v^{(sp-2)p^{i-1}}g_0) &= s(s+1)v^{(sp-2)p^{i+n-2}}g_{n-1} \end{aligned}$$

for $a = a(i, [i], s)$, which equals $s(s+1)u^{p^2+p-2}v^{(sp-2)p^{i+n-2}}g_0$ by the relation $u^{p+2}g_{n-1} = u^{p^2+2p}g_0$. This relation follows from (3.1.4) and $uh_{n-1} = u^p h_0$ given by $d(v)$. □

3.4 The summands C_G and C_K

We study the action of the connecting homomorphism δ by use of the Massey product. We notice that this is also shown by use of P^0-operation considered in the previous section, but we use the Massey product for the sake of simplicity.

Lemma 3.4.1. *The condition* $(3.2.6)_{(i,j,s)}$ *holds for* $(i,j) \in G \cup K$.

Proof. We consider the element $(1/u^{a(i,j,s)})(x_i^s h_j)$ the Massey product $\langle sx_{i-1}^{sp-1}/u^{a_{i-1}}, h_{[i-1]}, h_j\rangle$. Then, $\delta'_{(i,j,s)}(x_i^s h_j) = \delta\langle sx_{i-1}^{sp-1}/u^{a_{i-1}}, h_{[i-1]}, h_j\rangle = \langle s\delta(x_{i-1}^{sp-1}/u^{a_{i-1}}), h_{[i-1]}, h_j\rangle$, which equals $-\langle sv^{sp-2}h_{n-1}, h_0, h_0\rangle = -sv^{(s-2)p}k_{n-1}$ if $i=1$, and $-\langle sv^{(sp^2-p-1)p^{i-2}}h_{[i-2]}, h_{[i-1]}, h_j\rangle = \begin{cases} -sv^{(sp^2-p-1)p^{i-2}}k_{j-1} & j = [i-1], \\ -2sv^{(sp^2-p-1)p^{i-2}}g_j & j = [i-2] \end{cases}$ otherwise. Here, we note that $\langle h_i, h_{i+1}, h_i\rangle = 2g_i$. □

3.5 The summand $V_{(0,n-2)}$

Consider the elements $c_i = u^{p^i}h_{n-1+i}$ and $c'_i = u^{p^{i+1}}h_i$ of $\mathrm{Ext}^1 A$. The elements have internal degrees $|c_i| = |c'_i| = p^i e(n)q$ for $q = 2p-2$, and satisfy

$$c_i = c'_i, \quad c_i c_{i+1} = 0, \quad h_{n+i}c_i = 0 \quad \text{and} \quad h_{i+1}c_i = h_{i+1}c'_i = 0.$$

We consider the cochains $\overline{w}_k = u^{e(k-1)}ct_k^{p^{n-1}}$ of the cobar complex $\Omega^1_\Gamma A$. Then,

$$\overline{w}_k = -\overline{w}_{k-1}^p\eta_R(v) + u^{pe(k-2)}v^{p^{k-1}}ct_{k-1} + u^{p^k+pe(k-2)}ct_k \tag{3.5.1}$$

for $k > 1$ by (3.3.1). Let w_k be a cochain of the cobar complex $\Omega^1_\Gamma A$ defined inductively by:

$$\begin{aligned} w_1 &= t_1^{p^{n-1}} - u^{p-1}t_1 = -\overline{w}_1 + u^{p-1}ct_1 \quad \text{and} \\ w_k &= w_{k-1}^p\eta_R(v) + (-1)^k u^{pe(k-2)}v^{p^{k-1}}ct_{k-1} \end{aligned} \tag{3.5.2}$$

and put

$$\begin{aligned} m'_k &= -\textstyle\sum_{i=1}^{k-1}(-1)^i u^{p^{i-1}}w_{k-i}^{p^i} \otimes \overline{w}_i \quad \text{and} \\ m_k &= u^{p^{k-1}}w_k + \textstyle\sum_{i=1}^{k-1}(-1)^i u^{p^{i-1}}v^{p^i e(k-i)}\overline{w}_i. \end{aligned} \tag{3.5.3}$$

Lemma 3.5.4. $d(v^{e(k)}) = m_k$. *Besides,* $d(w_k) = m'_k$ *if* $k \le n$.

Proof. We prove the lemma inductively. Since $d(v) = uw_1 = m_1$, we see the case for $k = 1$. Indeed, $m'_1 = 0$. Suppose that the equalities hold for $k-1$. Then, we compute by (3.3.3), (3.5.1) and (3.5.2),

$$\begin{aligned} d(v^{e(k)}) &= d(v^{pe(k-1)})\eta_R(v) + v^{pe(k-1)}d(v) \\ &= \left(u^{p^{k-1}}w_{k-1}^p + \textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}v^{p^{i+1}e(k-1-i)}\overline{w}_i^p\right)\eta_R(v) - uv^{pe(k-1)}\left(\overline{w}_1 - u^{p-1}ct_1\right) \\ &= u^{p^{k-1}}\left(w_k - (-1)^k u^{pe(k-2)}v^{p^{k-1}}ct_{k-1}\right) - uv^{pe(k-1)}\left(\overline{w}_1 - u^{p-1}ct_1\right) \\ &\quad + \textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}v^{p^{i+1}e(k-1-i)}\left(-\overline{w}_{i+1} + \left(u^{pe(i-1)}v^{p^i}ct_i + u^{p^{i+1}+pe(i-1)}ct_{i+1}\right)\right), \end{aligned}$$

which equals m_k, and similarly,

$$\begin{aligned} d(w_k) &= -\textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}w_{k-1-i}^{p^{i+1}} \otimes \overline{w}_i^p\eta_R(v) + uw_{k-1}^p \otimes (\overline{w}_1 - u^{p-1}ct_1) \\ &\quad +(-1)^k u^{pe(k-2)}\left(u^{p^{k-1}}w_1^{p^{k-1}} \otimes ct_{k-1} + v^{p^{k-1}}d(ct_{k-1})\right) \\ &= -\textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}w_{k-1-i}^{p^{i+1}} \otimes \left(-\overline{w}_{i+1} + \underline{u^{pe(i-1)}v^{p^i}ct_i + u^{p^{i+1}+pe(i-1)}ct_{i+1}}\right) \\ &\quad +uw_{k-1}^p \otimes (\overline{w}_1 - \underline{u^{p-1}ct_1}) \\ &\quad \underline{+(-1)^k u^{e(k-2)}\left(u^{p^{k-1}}w_1^{p^{k-1}} \otimes ct_{k-1} + v^{p^{k-1}}d(ct_{k-1})\right)} = m'_k \end{aligned}$$

Here, the underlined terms cancel each other if $k \le n$ by (3.5.2) and (3.3.1) with the relation $\Delta(cx) = T(c \otimes c)\Delta(x)$ for the switching map $T\colon \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma$. $\square$

We also introduce an element

$$\overline{c}_k = h_{n+k-1} - u^{(p-1)p^k}h_k \in \mathrm{Ext}^1 A.$$

Corollary 3.5.5. *For each $0 < k < n$, the Massey products $\mu_k = \langle u^{p^k}, \overline{c}_k, c_{k-1}, c_{k-2}, \dots, c_1, c_0\rangle$ and $\mu'_k = \langle \overline{c}_k, c_{k-1}, c_{k-2}, \dots, c_1, c_0\rangle$ are defined. In fact, the cocycles m_{k+1} and m'_{k+1} represent elements of the Massey products μ_k and μ'_k, respectively.*

In particular, we have

Corollary 3.5.6. *The Massey product $\langle u^{p^{n-3}}, \overline{c}_{n-3}, c_{n-4}, \dots, c_0\rangle \subset \mathrm{Ext}^1 A$ is defined and contains zero.*

Lemma 3.5.7. *The Massey product $\langle \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle \subset \mathrm{Ext}^2 A$ contains zero.*

Corollary 3.5.8. *The Massey product $\mu = \langle u^{p^{n-3}}, \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle$ is defined and contain an element whose leading term is $v^{e(n-2)}h_{n-2}$.*

Lemma 3.5.9. *The condition $(3.2.6)_{(i,j,s)}$ holds for $(i,j) = (0, n-2)$.*

Proof. If $p \nmid s(s-1)$, it follows from the computation

$$\begin{aligned} d(v^s t_1^{p^{n-2}}) &\equiv suv^{s-1}t_1^{p^{n-1}} \otimes t_1^{p^{n-2}} + \tbinom{s}{2}u^2 t_1^{2p^{n-1}} \otimes t_1^{p^{n-2}} \bmod (u^3) \\ d(suv^{s-1}ct_2^{p^{n-2}}) &\equiv s(s-1)u^2 t_1^{p^{n-1}} \otimes ct_2^{p^{n-2}} - suv^{s-1}t_1^{p^{n-1}} \otimes t_1^{p^{n-2}} \bmod (u^3). \end{aligned}$$

Suppose $s = tp^l + e(n-2)$ with $p \nmid t$ and $l > 0$. Let $\widetilde{\theta}$ denote an element of Corollary 3.5.8. We take a generator corresponding to $v^s h_{n-2}$ to be $v^{s-e(n-2)}\widetilde{\theta}$. We denote a representative of $\widetilde{\theta}$ by m, which is congruent to $v^{e(n-2)}t_1^{p^{n-2}} + uv^{pe(n-3)}ct_2^{p^{n-2}}$ modulo (u^2). Then, $d(v^{s-e(n-2)}m) = tu^{a_l}v^{s-e(n-2)-p^{l-1}}t_1^{p^{[l-1]}} \otimes m \equiv tu^{a_l}v^{s-p^{l-1}}t_1^{p^{[l-1]}} \otimes t_1^{p^{n-2}}$. This shows the case for $[l] \neq 0, n-2$.

For $[l] = 0$, the similar computation shows that $d(v^{s-e(n-2)}m) \equiv tu^{a_l}v^{s-p^{l-1}}(t_1^{p^{n-2}} \otimes t_1^{p^{n-2}} + uv^{-1}t_1^{p^{n-1}+p^{n-2}} \otimes t_1^{p^{n-2}} + uv^{-1}t_1^{p^{n-2}} \otimes ct_2^{p^{n-2}})$, which yields $v^{s-1-p^{l-1}}g_{n-2}$. For $[l] = n-2$, $\widetilde{\theta}h_{n-3} \in u^{e(n-2)}\langle h_{2n-4}, h_{2n-5}, \ldots, h_{n-2}, h_{n-3}\rangle = \{u^{e(n-2)+p^{n-3}}b_{2n-5}\}$ in $C(p^{n-2})$. Indeed, $u^{e(n-3)}t_n^{p^{n-3}}$ yields the equality by (3.3.1). $\square$

3.6 On the action of α_1 and β_1 on Greek letter elements

In this section, let H^*M for a $BP_*(BP)$-comodule M denote an Ext group $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, M)$. Consider the comodule $N_{k-1}(j) = BP_*/(I_{k-1} + (v_{k-1}^j))$ $(v_0 = p)$, and the connecting homomorphism $\delta_{k,j}$ associated to the short exact sequence $0 \to BP_*/I_{k-1} \xrightarrow{v_{k-1}^j} BP_*/I_{k-1} \to N_{k-1}(j) \to 0$. We abbreviate $\delta_{k,1}$ to δ_k. Here we consider the Greek letter elements of H^*BP_*/I_{n-1} defined by

$$\begin{aligned} \overline{\alpha}_t^{(n-1)} &= u^t \in H^0BP_*/I_{n-1} \quad \text{and} \\ \alpha_{(t/j)}^{(n)} &= \delta_{n,j}(v^t) \in H^1BP_*/I_{n-1} \quad \text{for } v^t \in H^0N_{n-1}(j) \end{aligned}$$

for $t > 0$, and

$$\alpha_1 \;=\; \delta_1(v_1) = h_0 \in H^1BP_* \quad \text{and} \quad \beta_1 = \delta_1\delta_2(v_2) = b_0 \in H^2BP_*.$$

Proposition 3.6.1. *The elements α_1 and β_1 act on the Greek letter elements as follows:*

$$\alpha_1\overline{\alpha}_t^{(n-1)} \neq 0 \in H^1BP_*/I_{n-1}, \quad \beta_1\overline{\alpha}_t^{(n-1)} \neq 0 \in H^2BP_*/I_{n-1};$$

and if the Greek letter elements $\alpha_{(sp^i/j)}^{(n)}$ has an internal degree greater than $2(p^n-1)(e(n-1)-1)$, then

$$\begin{aligned} &\alpha_1\alpha_{(sp^i/j)}^{(n)} \neq 0 \in H^2BP_*/I_{n-1} \text{ if } [i] \neq 0,\ p \nmid (s+1) \text{ or } p^2 \mid (s+1); \text{ and} \\ &\beta_1\alpha_{(sp^i/j)}^{(n)} \neq 0 \in H^3BP_*/I_{n-1} \text{ if } n \neq 5,\ [i] \neq 1 \text{ or } p \nmid (s+1). \end{aligned}$$

In order to prove this, we make a chromatic argument: Let N_k^0 denote the BP_*BP-comodule BP_*/I_k, and put $M_k^0 = v_k^{-1}N_k^0$. We denote the cokernel of the inclusion $N_k^0 \to M_k^0$ by N_k^1, so that $0 \to N_k^0 \to M_k^0 \xrightarrow{\psi} N_k^1 \to 0$ is an exact sequence. Let $\widetilde{\delta}_{k+1}\colon H^sN_k^1 \to H^{s+1}N_k^0$ be the connecting homomorphism associated to the short exact sequence. We notice that $N_k^1 = \mathrm{colim}_j N_k(j)$ with inclusion $\varphi_j\colon N_k(j) \to N_k^1$ given by $\varphi_j(x) = x/u^j$, and that the connecting homomorphism $\delta_{n,j}\colon H^sN_{n-1}(j) \to H^{s+1}N_{n-1}^0$ factorizes to $\widetilde{\delta}_n\varphi_j$.

Lemma 3.6.2. *For an element $x_i^s/u^j \in H^0N_{n-1}^1$ for $0 < j \le a_i$ ($j \le p^i$ if $s = 1$), α_1 and β_1 act on it as follows:*

$$\begin{array}{ll} x_i^s\alpha_1/u^j \neq 0 \in H^1N_{n-1}^1 & \text{if } [i] \neq 0,\ p \nmid (s+1) \text{ or } p^2 \mid (s+1)\text{; and} \\ x_i^s\beta_1/u^j \neq 0 \in H^2N_{n-1}^1 & \text{if } n \neq 5,\ [i] \neq 1 \text{ or } p \nmid (s+1). \end{array}$$

Proof. A change of rings theorem of Miller and Ravenel [7] shows that the module $H^sM_{n-1}^1$ is isomorphic to Ext^sB. By (3.1.5), we see that $x_i^sh_0/u \neq 0 \in \mathrm{Ext}^1B$ unless $[i] = 0$, $p \mid (s+1)$ and $p^2 \nmid (s+1)$. This shows the first non-triviality. Similarly, since we have shown that (3.2.4) is exact, we see that $x_i^s\beta_1/u \neq 0 \in \mathrm{Ext}^2B$ unless $n = 5$, $[i] = 1$ and $p \mid (s+1)$. □

Lemma 3.6.3. *Let ξ_1 denote α_1 or β_1, and $x \in H^0N_{n-1}^1$, and suppose that $x\xi_1$ has an internal degree greater than $2(p^{n-1}-1)(e(n-1)-1)$. If $x\xi_1 \in H^sN_{n-1}^1 \neq 0$, then $\widetilde{\delta}_n(x)\xi_1 \neq 0 \in H^{s+1}N_{n-1}^0$.*

Proof. It suffices to show that $x\xi_1$ is not in the image of $\psi_*\colon H^sM_{n-1}^0 \to H^sN_{n-1}^1$. Again the change of rings theorem shows that the module $H^sM_{n-1}^0$ is isomorphic to the module of Lemma 3.1.3 with substituting $n-1$ for n. Note that every generator of it except for ζ_{n-1} belongs to $H^sN_{n-1}^0$, and also is $u^{e(n-1)}\zeta_{n-1}$ (*cf.* [14]). It follows that every element of the image of ψ_* has an internal degree no greater than $2(e(n-1)-1)(p^{n-1}-1)$. Thus the lemma follows. □

Proof of Proposition 3.6.1. The module $H^sM_{n-1}^0$ contains a submodule $k_*\langle h_0\rangle$ if $s = 1$ and $k_*\langle b_0\rangle$ if $s = 2$. Therefore, the first two relations hold. The other relations follow from Lemmas 3.6.2 and 3.6.3. □

Proof of Theorem 3.1.11. Note that $\overline{\alpha}_t^{(3)} = \overline{\gamma}_t = v_3^t$, and we obtain the theorem from Proposition 3.6.1 at $n = 4$. □

Bibliography

[1] Y. Arita and K. Shimomura, *The chromatic E_1-term $H^1M_1^1$ at the prime* 3. Hiroshima Math. J. 26 (1996), 415–431.

[2] H.-W. Henn, *Centralizers of elementary abelian p-subgroups and mod-p cohomology of profinite groups*, Duke Math. J. **91** (1998), 561–585.

[3] H. Hirata and K. Shimomura, *The chromatic E_1-term $H^1M_2^1$ for an odd prime*, in preparation.

[4] D. Kraines, *Massey higher products*, Trans. Amer. Math. Soc. **124** (1966), 431–449.

[5] J. P. May, *Matric Massey products*, J. Alg. **12** (1969), 533–568.

[6] J. P. May, *A general algebraic approach to Steenrod operations, The Steenrod Algebra and its Applications*, Lecture Notes in Mathematics **168** (1970), 153–231.

[7] H. R. Miller and D. C. Ravenel, *Morava stabilizer algebras and the localization of Novikov's E_2-term*, Duke Math. J. , **44** (1977), 433-447.

[8] H. R. Miller, D. C. Ravenel, and W. S. Wilson, *Periodic phenomena in Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), 469–516.

[9] H. Nakai, *The chromatic E_1-term $H^0M_1^2$ for $p > 3$*, New York J. Math. **6** (2000), 21–54 (electronic).

[10] H. Nakai, *The structure of* $\mathrm{Ext}^0_{\mathrm{BP}_*\mathrm{BP}}(\mathrm{BP}_*, M_1^2)$ *for* $p = 3$, Mem. Fac. Sci. Kochi Univ. Ser. A Math. 23 (2002), 27– 44.

[11] D. C. Ravenel, *The cohomology of the Morava stabilizer algebras*, Math. Z. **152** (1977), 287–297.

[12] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math., **106** (1984), 351–414.

[13] D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, 1992.

[14] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.

[15] K. Shimomura, *On the Adams-Novikov spectral sequence and products of β-elements*, Hiroshima Math. J. **16** (1986), 209–224.

[16] K. Shimomura, *The chromatic E_1-term $H^1M_2^1$ and its application to the homology of the Toda-Smith spectrum $V(1)$*, J. Fac. Educ. Tottori Univ. (Nat. Sci.) 39 (1990), 63–83. Corrections to "The chromatic E_1-term $H^1M_2^1$ and its application to the homology of the Toda-Smith spectrum $V(1)$", J. Fac. Educ. Tottori Univ. (Nat. Sci.) 41 (1992), 7–11.

[17] K. Shimomura, *The chromatic E_1-term $H^0M_n^2$ for $n > 1$*, J. Fac. Educ. Tottori Univ. (Nat. Sci.) **39** (1990), 103–121.

[18] K. Shimomura, *The homotopy groups of L_2-localized Toda-Smith spectrum $V(1)$ at the prime 3*, Trans. Amer. Math. Soc. **349** (1997), 1821–1850.

[19] K. Shimomura, *The homotopy groups of the L_2-localized mod 3 Moore spectrum*, J. Math. Soc. of Japan **51** (2000), 65–90.

[20] K. Shimomura and H. Tamura, *Non-triviality of some compositions of β-elements in the stable homotopy of the Moore spaces*, Hiroshima Math. J. **16** (1986), 121–133.

[21] K. Shimomura and A. Yabe, *The homotopy groups $\pi_*(L_2S^0)$*, Topology **34** (1995), 261–289.

[22] K. Shimomura and X. Wang, *The homotopy groups $\pi_*(L_2S^0)$ at the prime 3*, Topology **41** (2002), 1183–1198.

[23] H. Toda, *On spectra realizing exterior parts of the Steenrod algebra*, Topology 10 (1971), 53-65.

Chapter 4

Generalized Bousfield lattices and a generalized retract conjecture

In [1], Bousfield studied a lattice (Bousfield lattice) on the stable homotopy category of spectra, and in [5], Hovey and Palmieri made the retract conjecture on the lattice. In this chapter we generalize the Bousfield lattice and the retract conjecture to the ones on a monoid. We also determine the structure of typical examples of them, which satisfy the generalized retract conjecture. In particular we give the structure of the Bousfield lattice of the stable homotopy category of harmonic spectra explicitly. This is joit work with Professor Shimomura and Yotaro Tatehara.

4.1 Introduction

Let $\mathcal{M}$ be a closed symmetric monoidal category with zero object, and consider an object M of it. We call the full subcategory $\langle M\rangle$ of $\mathcal{M}$ the *Bousfield class* of M if it consists of objects A of $\mathcal{M}$ such that $MA = 0$ by its monoidal structure. Then we have a partial order on Bousfield classes by $\langle M\rangle \leq \langle N\rangle$ if every object of $\langle N\rangle$ is an object of $\langle M\rangle$. Then the subcategories $\langle S\rangle$ and $\langle O\rangle$ of the unit S and the zero O are the greatest and the least ones in the order, respectively. We call the collection of all Bousfield classes a *Bousfield lattice*, and denote it by $\mathbb{B}(\mathcal{M})$. In a case where a Bousfield lattice is a set, the partial order introduces a lattice structure to it, and we may investigate it algebraically.

In a sense, the stable homotopy theory is analyzing stable homotopy categories (*cf.* [6]). A stable homotopy category is a symmetric monoidal category, and so we may consider its Bousfield lattice. In particular, T. Ohkawa [8] (*cf.* [2]) showed that the Bousfield lattice $\mathbb{B}$ of the stable homotopy category of spectra is a set, and then Iyengar and Krause [7] generalized it to a stable homotopy category.

In order to investigate a category, we sometimes classify special subcategories of it. From this viewpoint, we study a Bousfield lattice by classifying localizing subcategories (see [6]). Indeed, every Bousfield class is a localizing subcategory.

In [5], Hovey and Palmieri studied the Bousfield lattice $\mathbb{B}$ deeply. Furthermore, they proposed many conjectures on the structure of $\mathbb{B}$. Among them, there is the retract conjecture, which is one of our main topics. Dwyer and Palmieri [3] constructed a stable homotopy category, where the conjecture does not hold. So far, there seems no nontrivial category in which the conjecture holds. In this chapter, we give some examples of categories with the affirmative answer to the conjecture.

As stated above, a Bousfield lattice $\mathbb{B}(\mathcal{M})$ is a set in some cases. In this case, it is a monoid with multiplication compatible with its order. We introduce the notion of monoidal posets and define a functor β from a subcategory of commutative monoids to the category of monoidal posets in Section two. Then we define a Bousfield lattice of a monoid to be an object in the image of β, which is an analogy of Bousfield lattices of stable homotopy categories. In particular, $\mathbb{B}$ has not only a structure of a monoidal poset, but also a Bousfield lattice associated to $\mathbb{B}$ itself. In section three, we show analogous properties on a Bousfield lattice to those given by Hovey and Palmieri [5] including the following:

Conjecture 4.1.1 (Original retract conjecture [5, Conj. 3.12])**.** *Let h be the Bousfield class of the* mod p *Eilenberg-MacLane spectrum $H\mathbb{Z}/p$ in the Bousfield lattice $\mathbb{B}$. Then, there is a lattice isomorphism $r_*\colon \mathbb{B}/J(h) \to \mathbb{DL}$. Here, $J(h)$ is an ideal related to h (see Notation 4.3.1).*

We generalize it to generalized retract conjectures on a monoidally distributive poset (Conjectures 4.3.18 and 4.3.19) and show some facts relating to them. Section four is devoted to determine Bousfield lattices obtained from principal ideal domains, and to show the conjecture true for them. In section five, we study about Bousfield lattices of stable homotopy categories of Bousfield localized spectra, and construct isomorphisms between the Bousfield lattice and a Bousfield lattice given in section four. In particular, we have the following:

Theorem 4.1.2. *The generalized retract conjectures holds on the stable homotopy category of harmonic spectra.*

One of our final goals is to determine the lattice structure of $\mathbb{B}$, which seems difficult so much. In the last section, we propose problems on the functor β, whose answers may help us to understand the Bousfield lattice $\mathbb{B}$. We expect that these problems give us hints to reach the goal.

4.2 Monoidal posets and Bousfield lattices

Let M be commutative monoid with unit 1. We call M a *monoid with* 0 if M admits an element $0 \in M$ such that $0 \cdot x = 0 = x \cdot 0$ for any $x \in M$.

A typical example of it is a commutative ring ignoring addition. We denote by $\mathcal{M}_0$ the category consisting of commutative monoids with 0 and monoid homomorphisms preserving zero.

For $M \in \mathcal{M}_0$, $\beta(M)$ denotes a set consisting of subsets

$$\langle x \rangle = \{y \in M : xy = 0\}$$

of M for $x \in M$.

Lemma 4.2.1. *$\beta(M)$ for $M \in \mathcal{M}_0$ is also a monoid with 0 with inherited multiplication. Therefore, we have the canonical epimorphism $M \to \beta(M)$ in $\mathcal{M}_0$.*

Proof. Define a multiplication of $\beta(M)$ by $\langle x \rangle \langle y \rangle = \langle xy \rangle$. We verify it well defined as follows: Assume that $\langle x_0 \rangle = \langle x_1 \rangle$ and $\langle y_0 \rangle = \langle y_1 \rangle$. Then

$$\begin{aligned} zx_0y_0 = 0 \quad &\Leftrightarrow \quad zx_1y_0 = 0 \quad \text{by } \langle x_0 \rangle = \langle x_1 \rangle \\ &\Leftrightarrow \quad zx_1y_1 = 0 \quad \text{by } \langle y_0 \rangle = \langle y_1 \rangle, \end{aligned}$$

and $\langle x_0y_0 \rangle = \langle x_1y_1 \rangle$. The elements $\langle 1 \rangle$ and $\langle 0 \rangle$ are the unit and the zero elements. □

Remark 4.2.2. We notice that $\beta(R) = \mathbb{Z}/2$ if R is a domain.

Lemma 4.2.3. *Let M be a monoid with 0. Then $\beta(M)$ admits a partial order '$\leq$' on M defined by $\langle x \rangle \leq \langle y \rangle$ if $\langle x \rangle \supset \langle y \rangle$. Besides, $\langle 1 \rangle$ and $\langle 0 \rangle$ are the greatest and the least elements, respectively.*

Proof. This is trivial since $\langle 1 \rangle = \{0\}$ and $\langle 0 \rangle = M$. □

By the lemma, a commutative monoid $\beta(M)$ has also a poset structure. Then we define the following notion by taking its crucial properties.

Definition 4.2.4. A *monoidal poset* $P = (P, \leq, \cdot, 1, 0)$ is defined by the following data.

(1) $(P, \cdot, 1, 0)$ is a monoid with 0.

(2) $(P, \leq)$ is a poset.

(3) The following are equivalent.

   (a) $x \leq y$.

   (b) $cy = 0$ for $c \in P$ implies $cx = 0$.

A *monoidal poset map* $f : P \to P'$ is an order preserving monoid homomorphism with $f(0) = 0$.

Lemma 4.2.3 implies the following.

Corollary 4.2.5. *$\beta(M)$ for $M \in \mathcal{M}_0$ is a monoidal poset with $1 = \langle 1 \rangle$ and $0 = \langle 0 \rangle$.*

Lemma 4.2.6. *Let M be a monoidal poset. Then, $\beta(M) = M$ as monoidal posets.*

Remark 4.2.7. A monoidal poset seems a lattice, but unfortunately it is not true. Indeed, we have an example: Consider a monoidal poset $M = \{1, x_i, y_i, w, 0 \colon i = 1, 2\}$ with multiplication

1	x_1	x_2	y_1	y_2	w
x_1	w	w	0	w	0
x_2	w	w	w	0	0
y_1	0	w	0	0	0
y_2	w	0	0	0	0
w	0	0	0	0	0

Then, the join of y_1 and y_2 does not exist.

Let $\mathcal{MP}$ denote the category of monoidal posets and monoidal poset maps. Then $\mathcal{MP} \subset \mathcal{M}_0$.

Lemma 4.2.8. *Let M be a monoidal poset. Then, $xz \leq yw$ if $x \leq y$ and $z \leq w$. In particular, if $x \leq y$, then $xz \leq yz$ for any z.*

Proposition 4.2.9. *The category $\mathcal{MP}$ admits direct products.*

Proof. Let $\{M_\lambda\}$ be a family of monoidal posets. Then, we have a direct product $\prod_\lambda M_\lambda$ of monoids. Consider an order '$\leq$' on $\prod_\lambda M_\lambda$ defined by $(x_\lambda) \leq (y_\lambda)$ if $(c_\lambda)(y_\lambda) = (0)$ implies $(c_\lambda)(x_\lambda) = (0)$. It is straightforward to verify this is the desired direct product. ☐

Lemma 4.2.10. *Let $\{M_\lambda\}$ be a family of monoidal posets. Then, $\langle x_\lambda \rangle \leq \langle y_\lambda \rangle$ for all λ if and only if $\langle (x_\lambda) \rangle \leq \langle (y_\lambda) \rangle$. Here, $\langle x_\lambda \rangle, \langle y_\lambda \rangle \in \beta(M_\lambda)$ and $\langle (x_\lambda) \rangle, \langle (y_\lambda) \rangle \in \beta(\prod_\lambda M_\lambda)$.*

Proof. Assume that $\langle x_\lambda \rangle \leq \langle y_\lambda \rangle$ for any λ. Then

$$\begin{aligned}
(c_\lambda)(y_\lambda) = 0 \quad &\Rightarrow \quad c_\lambda y_\lambda = 0 \text{ for any } \lambda \\
&\Rightarrow \quad c_\lambda x_\lambda = 0 \text{ for any } \lambda \ (\because \langle x_\lambda \rangle \leq \langle y_\lambda \rangle) \\
&\Rightarrow \quad (c_\lambda)(x_\lambda) = 0,
\end{aligned}$$

Conversely, suppose that $\langle (x_\mu) \rangle \leq \langle (y_\mu) \rangle$. Then, for any λ,

$$\begin{aligned}
y_\lambda c_\lambda = 0 \quad &\Rightarrow \quad (y_\lambda)(c_\lambda)_0 = 0 \\
&\Rightarrow \quad (x_\lambda)(c_\lambda)_0 = 0 \ (\because \langle (x_\mu) \rangle \leq \langle (y_\mu) \rangle) \\
&\Rightarrow \quad x_\lambda c_\lambda = 0
\end{aligned}$$

in M_λ, where $(c_\lambda)_0$ denotes an element (x_μ) such that $x_\lambda = c_\lambda$ and $x_\mu = 0$ for $\mu \neq \lambda$. ☐

Corollary 4.2.11. *Let $\{M_\lambda\}$ be a family of monoidal posets. Define an order $\leq'$ on the set $\prod_\lambda M_\lambda$ by $(x_\lambda) \leq' (y_\lambda)$ if $x_\lambda \leq y_\lambda$ for all λ. Then it is equivalent to the order in the proof of Proposition 4.2.9.*

Corollary 4.2.12. *Let $\{M_\lambda\}$ be a family of monoidal posets. Then, $\bigvee_\mu (x^\mu_\lambda) = (\bigvee_\mu x^\mu_\lambda)$ for any subset $\{(x^\mu_\lambda)\}_\mu \subset \prod_\lambda M_\lambda$.*

Proof. Since $(x^\mu_\lambda) \leq (\bigvee_\mu x^\mu_\lambda)$ for all μ, $\bigvee_\mu (x^\mu_\lambda) \leq (\bigvee_\mu x^\mu_\lambda)$. If $(x^\mu_\lambda) \leq (z_\lambda)$, then $x^\mu_\lambda \leq z_\lambda$, and so $\bigvee_\mu x^\mu_\lambda \leq z_\lambda$, that is, $(\bigvee_\mu x^\mu_\lambda) \leq (z_\lambda)$. Therefore, $\bigvee_\mu (x^\mu_\lambda) = (\bigvee_\mu x^\mu_\lambda)$ by definition. □

We call an epimorphism $f\colon M \to N$ of $\mathcal{M}_0$ *strong* if $f(x) = 0$ if and only if $x = 0$.

We define a map $\beta(f)\colon \beta(M) \to \beta(N)$ by sending $\langle x \rangle$ to $\langle f(x) \rangle$.

Lemma 4.2.13. *For a strong epimorphism $f\colon M \to N$, the map $\beta(f)$ is not only a monoidal poset map but also a strong epimorphism.*

Proof. Since f is a strong epimorphism, $c \cdot f(x) = 0 \Leftrightarrow f(c') \cdot f(x) = 0 \Leftrightarrow f(c' \cdot x) = 0 \Leftrightarrow c' \cdot x = 0$ for an element c' such that $f(c') = c$. This shows that $\langle x \rangle = \langle y \rangle$ implies $\langle f(x) \rangle = \langle f(y) \rangle$. It is easy to see that $\beta(f)$ is a strong epimorphism. □

We also consider the subcategories $\mathcal{M}$ and $\mathcal{MP}^{epi}$ of $\mathcal{M}_0$ and $\mathcal{MP}$, respectively, obtained by restricting morphisms to strong epimorphisms.

Corollary 4.2.14. *The operation β above defines a functor $\beta\colon \mathcal{M} \to \mathcal{MP}^{epi} \subset \mathcal{M}$.*

By the above argument, we redefine Bousfield lattices as follows. The definition is one of our main topics in this chapter.

Definition 4.2.15. For a monoid $M \in \mathcal{M}$ we call a monoidal poset $\beta(M)$ the *Bousfield lattice associated to M*.

In earlier papers, a Bousfield lattice is made from a closed symmetric monoidal category with a zero object. However, its set theoretic confusion complicates our argument too much. Our new definition settles this problem, and the following proposition says that this argument is consistent.

Proposition 4.2.16. *The Bousfield lattice $\mathbb{B}$ of the stable homotopy category of spectra is a Bousfield lattice in the sense of our definition.*

Proof. By forgetting the ordering on $\mathbb{B}$, we regard $\mathbb{B}$ as a monoid with $1 = \langle S \rangle$ and $0 = \langle * \rangle$. Then it is clear that $\beta(\mathbb{B}) = \mathbb{B}$. □

Proposition 4.2.17. *The functor β satisfies the following:*

(1) $\beta(\prod_\lambda M_\lambda) = \prod_\lambda \beta(M_\lambda)$.

(2) $\beta\beta(M) = \beta(M)$.

Proof. (1) Let $\{p_\lambda \colon \beta(\prod_\lambda M_\lambda) \to \beta(M_\lambda)\}$ be a family of epimorphisms defined by $\langle(x_\lambda)\rangle \mapsto \langle x_\lambda\rangle$, and $\{f_\lambda \colon W \to \beta(M_\lambda)\}$ a family of poset maps. We notice that p_λ is well defined by Lemma 4.2.10. For an element $w \in W$, we take an element $w_\lambda \in W_\lambda$ so that $f_\lambda(w) = \langle w_\lambda\rangle$, and define $g \colon W \to \beta(\prod_\lambda M_\lambda)$ by $g(w) = \langle(w_\lambda)\rangle$. Then g is also a well defined poset map by Lemma 4.2.10 and

$$p_\lambda g(w) = p_\lambda(\langle(w_\lambda)\rangle) = \langle w_\lambda\rangle = f_\lambda(w).$$

Suppose that there is another poset map $g' \colon W \to \beta(\prod_\lambda M_\lambda)$ satisfying $p_\lambda g'(w) = f_\lambda(w)$ for $w \in W$, and g' assigns w to $\langle(w'_\lambda)\rangle$. Then

$$\begin{array}{rcl} p_\lambda g'(w) = f_\lambda(w) \text{ for any } \lambda & \Leftrightarrow & \langle w'_\lambda\rangle = \langle w_\lambda\rangle \text{ for any } \lambda \\ & \Leftrightarrow & \langle(w'_\lambda)\rangle = \langle(w_\lambda)\rangle \text{ (by Lemma 4.2.10)} \\ & \Leftrightarrow & g'(w) = g(w). \end{array}$$

Therefore, $\beta(\prod_\lambda M_\lambda)$ is the product $\prod_\lambda \beta(M_\lambda)$.

(2) is seen by Lemma 4.2.6. $\square$

4.3 Retract conjecture

From now on, we assume that every monoidal poset considered is a complete lattice.

Since a monoidal poset M is a sup-lattice with the least element $0 = \langle 0\rangle$, M is a bounded lattice.

Notation 4.3.1. *For a monoidal poset M, we define the following notations.*

$$\begin{array}{rcl} a_M(x) & := & \bigvee\{y \in M \colon xy = 0\} \text{ for } x \in M, \\ BA(M) & : = & \{x \in \beta(X) \colon x \vee a(x) = 1\}, \\ DL(M) & := & \{x \in M : x^2 = x\}, \\ r_M(x) & := & \bigvee\{w \in DL(M) \colon w \leq x\} \text{ for } x \in M, \\ J_M(x) & := & \{y \in M \colon y \leq x \cdot a_M(x)\} \text{ for } x \in M, \\ N(M) & := & \{x \in M \colon x^n = 0 \text{ for some } n \geq 1\}, \\ A(M) & := & \{x \in M \colon r_M(x) = 0\}. \end{array}$$

We will omit M from notations, if M is clear from the context.

It is well known that the subposet $DL(M)$ is also a complete lattice. Indeed the following holds.

Proposition 4.3.2. *$DL(M)$ is closed under arbitrary joins.*

Proof. By Lemma 4.2.8, $(\bigvee_{\lambda\in\Lambda} x_\lambda)^2 \leq (\bigvee_{\lambda\in\Lambda} x_\lambda)$. Suppose that x_λ is in DL for $\lambda \in \Lambda$. Then, $x_\lambda = x_\lambda^2 \leq (\bigvee_{\lambda\in\Lambda} x_\lambda)^2$, and so $\bigvee_{\lambda\in\Lambda} x_\lambda \leq (\bigvee_{\lambda\in\Lambda} x_\lambda)^2$. $\square$

Lemma 4.3.3. *In DL(M), the meet of x and y is xy.*

Proof. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, if $x \wedge y \in DL(M)$ then $x \wedge y \leq xy$. $\square$

Remark 4.3.4. $DL(M)$ is not always sublattice of M by Lemma 4.3.3.

For investigating the original Bousfield lattice $\mathbb{B}$, the operations r and a play important roles (see [5]). Hereafter we try to give their properties analogously on monoidal posets.

Proposition 4.3.5. *Let M be a monoidal poset, and $r = r_M : M \to M$ be the map defined in Notation 4.3.1.*

(1) *r is order-preserving i.e. $x \le y$ implies $r(x) \le r(y)$.*

(2) *$r(x)^2 = r(x)$ and $r^2(x) = r(x)$ for $x \in M$.*

(3) *$r(x) \le x^n$ for any $n \ge 1$.*

(4) *$r(xy) = r(x)r(y) = r(x \wedge y)$ for $x, y \in M$.*

Proof. (1) is trivial, and (2) follows from Proposition 4.3.2. For (3), $r(x) \le x$ by definition, and we have $r(x) = r(x)^n \le x^n$.

Since $r(x)r(y) \le xy$ and $r(x)r(y) \in DL(M)$, we have $r(x)r(y) \le r(xy)$. We also see $r(x \wedge y) \le r(x)r(y)$, since $r(x \wedge y) \le r(x)$ and $r(x \wedge y) \le r(y)$. Therefore, $r(xy) \le r(x \wedge y) \le r(x)r(y) \le r(xy)$, and obtain (4). □

The behavior of the map r is the same as the one on $\mathbb{B}$, but not that of the operation a. Indeed, for any $x \in M$ and $\{y_\lambda\}_\lambda \subset M$, the relation $x(\bigvee_\lambda y_\lambda) \ge \bigvee_\lambda (xy_\lambda)$ is not always an equality. To make the operator a have good properties, we introduce a following notion.

Definition 4.3.6. A monoidal poset M is a *monoidally distributive* poset if M satisfies that $x(\bigvee_\lambda y_\lambda) = \bigvee_\lambda (xy_\lambda)$ for any $x \in M$ and $\{y_\lambda\}_\lambda \subset M$.

Remark 4.3.7. $DL(M)$ is a distributive lattice if M is a monoidally distributive poset by Lemma 4.3.3.

In the same way as [5], we have

Proposition 4.3.8. *Let M be a monoidally distributive poset. Then,*

(1) *$a(-)$ is order-reversing.*

(2) *$xy = 0$ if and only if $x \le a(y)$.*

(3) *$aa(x) = x$.*

Lemma 4.3.9. *Let M be a monoidally distributive poset. Fix $c \in M$ such that $c^n = 0$ for a positive integer n. Then, for any $x \in M$, $(x \vee c)^n \le x$ and $r(x \vee c) = r(x)$.*

Proof. Under the assumption, we compute

$$\begin{aligned}(x \vee c)^n &= x^n \vee x^{n-1}c \vee \cdots \vee xc^{n-1} \\ &= x(x^{n-1} \vee x^{n-2}c \vee \cdots \vee c^{n-1}) \le x\end{aligned}$$

for any $x \in M$. So, if $z \le x \vee c$ for $z \in DL(M)$, then $z \le x$. Thus, $r(x \vee c) = r(x)$ by definition of r. □

Proposition 4.3.10. *Let M be a monoidally distributive poset. Then $J_M(x) \subset N(M) \subset A(M)$ for any $x \in M$.*

Proof. Since $(x \cdot a_M(x))(x \cdot a_M(x)) \leq xa_M(x) = 0$ by Proposition 4.3.8(2), we have $J_M(x) \subset N(M)$. Suppose that $x^n = 0$, then $r(x) = r(x)^n = r(x^n) = r(0) = 0$ by Proposition 4.3.5 (4). So we have $N(M) \subset A(M)$. □

Proposition 4.3.11. *Let M_λ be a monoidal poset for any $\lambda \in \Lambda$. Then,*

(1) *$r((x_\lambda)) = (r(x_\lambda))$ for any $(x_\lambda) \in \prod_\lambda M_\lambda$.*

(2) *r preserves arbitrary joins on M_λ for any $\lambda \in \Lambda$ if and only if r preserves arbitrary joins on $\prod_\lambda M_\lambda$*

Proof. (1) is given by Corollary 4.2.12.

(2) Suppose that r preserves arbitrary joins on M_λ for any $\lambda \in \Lambda$. Then, for $\{(x^\mu_\lambda)\}_\mu \subset \prod_\lambda M_\lambda$,

$$\begin{aligned} r(\textstyle\bigvee_\mu (x^\mu_\lambda)) &= r((\textstyle\bigvee_\mu x^\mu_\lambda)) \text{ (by Corollary 4.2.12)} \\ &= (r(\textstyle\bigvee_\mu x^\mu_\lambda)) \text{ (by (1))} \\ &= (\textstyle\bigvee_\mu r(x^\mu_\lambda)) \\ &= \textstyle\bigvee_\mu (r(x^\mu_\lambda)) \text{ (by Corollary 4.2.12).} \end{aligned}$$

Therefore, r preserves arbitrary joins on $\prod_\lambda M_\lambda$.

Conversely, if r preserves arbitrary joins on $\prod_\lambda M_\lambda$, then

$$\begin{aligned} (r(\textstyle\bigvee_\mu x^\mu_\lambda)) &= r((\textstyle\bigvee_\mu x^\mu_\lambda)) \text{ (by (1))} \\ &= r(\textstyle\bigvee_\mu (x^\mu_\lambda)) \text{ (by Corollary 4.2.12)} \\ &= \textstyle\bigvee_\mu (r(x^\mu_\lambda)) \\ &= (\textstyle\bigvee_\mu r(x^\mu_\lambda)) \text{ (by Corollary 4.2.12).} \end{aligned}$$

It follows that r preserves arbitrary joins on M_λ for any $\lambda \in \Lambda$ as desired. □

Remark 4.3.12. We notice that M_λ is a monoidally distributive poset for any $\lambda \in \Lambda$ if and only if $\prod_{\lambda\in\Lambda} M_\lambda$ is a monoidally distributive poset. Indeed, if M_λ is a monoidally distributive poset for any $\lambda \in \Lambda$, then $(c_\lambda)(\bigvee_\mu (x^\mu_\lambda)) = (c_\lambda)(\bigvee_\mu x^\mu_\lambda) = (c_\lambda(\bigvee_\mu x^\mu_\lambda)) = (\bigvee_\mu c_\lambda x^\mu_\lambda) = \bigvee_\mu (c_\lambda x^\mu_\lambda)$ for $(c_\lambda) \in \prod_\lambda M_\lambda$ and $\{(x^\mu_\lambda)\}_\mu \subset \prod_\lambda M_\lambda$ by Corollary 4.2.12. Thus, $\prod_\lambda M_\lambda$ is a monoidally distributive poset. Conversely, if $\prod_\lambda M_\lambda$ is a monoidally distributive poset, then $(c_\lambda(\bigvee_\mu x^\mu_\lambda)) = (c_\lambda)(\bigvee_\mu x^\mu_\lambda) = (c_\lambda)(\bigvee_\mu (x^\mu_\lambda)) = \bigvee_\mu (c_\lambda x^\mu_\lambda) = (\bigvee_\mu c_\lambda x^\mu_\lambda)$ by Corollary 4.2.12. Therefore, M_λ is a monoidally distributive poset for any $\lambda \in \Lambda$ by Lemma 4.2.10.

Recall that an ideal I of a poset is any subset of M such that:

(1) If $x \in I$, and $y \leq x$, then $y \in I$, and

(2) For $x, y \in I$, there is an element $z \in I$ such that $x \leq z$ and $y \leq z$.

Suppose that a monoidal poset M is an ordinary lattice. Then, an ideal of M is also an ideal as a lattice, and for an ideal I, M/I is the lattice of equivalent classes under the equivalence relation defined by

$$x \sim y \quad \text{if and only if } x \vee c = y \vee c \text{ for some } c \in I \tag{4.3.13}$$

with order given by $[x] \leq [y] \Leftrightarrow x \vee c \leq y \vee c$ for some $c \in I$. We notice that M/I is complete if M and I are complete. If M is monoidally distributive, then M/I has the multiplication $[x][y] := [xy]$. Indeed, if $x \vee i = x' \vee i$ and $y \vee j = y' \vee j$ for $x, x', y, y' \in M$ and $i, j \in I$, then $(x \vee i)(y \vee j) = (x' \vee i)(y' \vee j)$ turns into

$$\begin{aligned} xy \vee (x \vee i)j \vee (y \vee j)i &= x'y' \vee (x' \vee i)j \vee (y' \vee j)i \\ &= x'y' \vee (x \vee i)j \vee (y \vee j)i. \end{aligned}$$

Since $(x \vee i)j \vee (y \vee j)i \in I$, the multiplication is well defined.

Remark 4.3.14. M/I is not always a monoidal poset. Indeed, we have an example: Let $M = \{1, x, y, 0\}$ be a monoidal poset with multiplication $x^2 = x, xy = 0, y^2 = 0$. Then, for the ideal $I = \{y, 0\}$, $M/I = \{1, x, 0\}$ and $\beta(M/I) = \{1, 0\}$. Since $M/I \neq \beta(M/I)$, M/I is not a monoidal poset by Lemma 4.2.6.

Lemma 4.3.15. *Let M be a monoidally distributive poset. Then, $N(M)$ is an ideal of M and $J_M(x)$ is a principal ideal of M for any $x \in M$.*

Proof. Suppose that $x^n = 0$ and $y^m = 0$. Then, $(x \vee y)^{n+m} = \bigvee_{a+b=n+m} x^a y^b$. Since if $a < n$ then $b \geq m$, $(x \vee y)^{n+m} = 0$. So $N(M)$ is an ideal of M. By definition, $J_M(x)$ is a principal ideal of M. □

Here, consider the following correspondence:

$$r_* \colon M/I \to DL(M); [x] \mapsto \{r(y) \colon y \in [x]\}$$

We notice that if r_* is a mapping (*i.e.* a single-valued mapping), then it is a surjection.

Theorem 4.3.16. *Let M be a monoidally distributive poset and I an ideal in M.*

(1) *If I is contained in N, then r_* is a mapping.*

(2) *If r_* is a mapping, then $I \subset A$.*

(3) *If r_* is an injection, then $I = A$.*

(4) *If r_* is an injection and $I \subset N$, then:*

   (a) *For any x and y in M, $r(x \vee y) = r(x) \vee r(y)$ holds. In particular, if I is a principal ideal, then r preserves arbitrary joins.*

   (b) *For any $x \in M$, there exists an integer n such that $x^n = r(x)$.*

Proof. (1) If $x \vee c = y \vee c$ for $x, y \in M$ and $c \in I \subset N$, then $r(x) = r(y)$ by Lemma 4.3.9.

(2) For $x \in I$, $[x] = 0 = [0]$ in M/I, and so $r(x) = r_*([x]) = r_*([0]) = r(0) = 0$. Thus, $x \in A$.

(3) For $x \in A$, $r_*([x]) = r(x) = 0 = r_*([0])$. It follows that $[x] = [0]$, since r_* is an injection, which implies $x \in I$. So we obtain $A = I$ by (2).

(4) For $x \in M$, $r_*([x]) = r(x) = r^2(x) = r_*([r(x)])$ and $[x] = [r(x)]$, since r_* is an injection. So we have an element $c_x \in N$ such that $x \vee c_x = r(x) \vee c_x$, and then:

(a) Since $x \vee y \vee c_x \vee c_y = r(x) \vee r(y) \vee c_x \vee c_y$, $r(x \vee y) = r(x) \vee r(y)$ by Lemma 4.3.9. Suppose that I is a principal ideal and take a generator m of I. Then, $(\bigvee_\lambda x_\lambda) \vee m = (\bigvee_\lambda r(x_\lambda)) \vee m$ for any subset $\{x_\lambda\}_\lambda \subset M$. Therefore $r(\bigvee_{\lambda\in\Lambda} x_\lambda) = \bigvee_{\lambda\in\Lambda} r(x_\lambda)$ by Lemma 4.3.9.

(b) Since there exists an integer n such that $c_x^n = 0$,

$$x^n \leq (x \vee c_x)^n = (r(x) \vee c_x)^n \leq r(x).$$

by Lemma 4.3.9. □

Hovey and Palmieri introduced a map $r_* \colon M/J(h) \to DL$, and proposed Conjecture 1.1 in the introduction. Here, we generalize the map to our setting.

Lemma 4.3.17. *The map $r_M \colon M \to M$ for a monoidal poset M factors through $DL(M)$. Furthermore, it induces the map $r_* \colon M/J_M(y) \to DL(M)$ for $y \in M$ assigning the class $[x]$ to $r_M(x)$.*

Proof. The former statement follows from Proposition 4.3.5(2), and the latter from Proposition 4.3.10 and Proposition 4.3.16(1). □

By Theorem 4.3.16, we see that $J(h) = A$ if Conjecture 4.1.1 holds. This makes us conjecture the following:

Conjecture 4.3.18 (Generalized retract conjecture 1 (GRC1))**.** *Let M be a monoidal poset. If M is a complete lattice and is monoidally distributive, and if $A = A(M)$ is an ideal of M, then $r_* \colon M/A \to DL$ is a lattice isomorphism.*

Conjecture 4.3.19 (Generalized retract conjecture 2 (GRC2))**.** *Let M be a monoidal poset. If M is a complete lattice and monoidally distributive, then $r_* \colon M/N \to DL(M)$ is a lattice isomorphism.*

By Theorem 4.3.16 (3), we see the following:

Corollary 4.3.20. *GRC2 implies GRC1.*

Example 4.3.21. *Consider the monoidal poset $M = \beta(\mathbb{Z}/2^m\mathbb{Z})$. Then,*

$$\begin{aligned} M &= \{1, 2, 2^2, \cdots, 2^{m-1}, 2^m = 0\}, \\ DL(M) &= \{1, 0\} \ \textit{and} \\ N(M) &= \{2, 2^2, \cdots, 2^{m-1}, 0\}. \end{aligned}$$

And so $M/N(M) \cong DL(M)$. That is, GRC2 holds on $\beta(\mathbb{Z}/2^m\mathbb{Z})$.

Theorem 4.3.22. *For a monoidally distributive poset M, the following are equivalent.*

(1) *$r_*\colon M/N \to DL$ is an isomorphism.*

(2) *Any class $[x] \in M/N$ satisfies $[x^2] = [x]$.*

Proof. The statement (1) implies (2), since $r_*([x]) = r_*([x^2])$.

For the converse, it suffices to show that r_* is injective. If $[x^2] = [x]$, then $[x] = [x^n]$ for any $n > 0$ by induction. So, we have an element $c_x \in N$ for each $x \in M$ such that

$$x \vee c_x = x^n \vee c_x \text{ for any } n > 0. \tag{4.3.23}$$

Since $c_x \in N$, we have an integer $L = L(x) > 0$ such that $c_x^L = 0$. Then

$$x^L \leq (x \vee c_x)^L = (x^n \vee c_x)^L \leq x^n$$

for any $n > 0$ by Lemma 4.3.9. In particular, $x^L = (x^L)^2$ and so

$$x^{L(x)} = r(x) \tag{4.3.24}$$

by Proposition 4.3.5.

Now suppose that $r_*([x]) = r_*([y])$. Then $r(x) = r(y)$, and $x^{L(x)} = y^{L(y)}$ by (4.3.24). By (4.3.23),

$$x \vee c_x \vee c_y = x^{L(x)} \vee c_x \vee c_y = y^{L(y)} \vee c_y \vee c_x = y \vee c_x \vee c_y$$

and $[x] = [y]$ by the definition (4.3.13). □

Furthermore, Proposition 4.3.11 leads us to the following.

Proposition 4.3.25. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of monoidally distributive posets. Then, the following are equivalent.*

(1) *GRC holds on M_λ for any $\lambda \in \Lambda$.*

(2) *GRC holds on $\prod M_\lambda$.*

Here, GRC is GRC1 or GRC2.

As an application, we extend a result of Dwyer and Palmieri:

Theorem 4.3.26 (Dwyer-Palmieri [3])**.** *There is a ring Λ such that the original retract conjecture does not hold on the derived category $D(\Lambda)$ of Λ.*

In the proof of it, Dwyer and Palmieri define Λ to be a truncated polynomial ring over a field k, and take $\langle k\rangle$ instead of $h = \langle H\mathbb{Z}/p\rangle$. Here $\langle k\rangle$ denotes a Bousfield class of a complex $\{X_i\}$ with $X_0 = k$, and $X_i = 0$ if $i \neq 0$. By a similar argument of Hovey and Palmieri in [5], if r_* is an isomorphism from $\mathbb{B}(D(\Lambda))/J(\langle k\rangle)$ to DL, then any Bousfield class $x \in \mathbb{B}(D(\Lambda))$ satisfies $x^2 = x^3$. They show the theorem by constructing a Bousfield class $y \in \mathbb{B}(D(\Lambda))$ such that $y > y^2 > \cdots > y^n > \cdots$. By Theorem 4.3.16, the existence of the class y implies further the following:

Theorem 4.3.27. *The map $r_*\colon \mathbb{B}(D(\Lambda))/N \to DL$ is not isomorphic.*

4.4 A Bousfield lattice associated to a quotient of PID

We abbreviate 'principal ideal domain' to 'PID'. Furthermore, we write x for $\langle x \rangle \in \beta(M)$, where no confusion arises.

Theorem 4.4.1. *Let P be a PID and put $q = p_0^{e_0} \cdots p_{m-1}^{e_{m-1}} \in P$ for prime elements p_i and integers $e_i > 0$. Let B denote a Bousfield lattice $\beta(P/qP)$. Then,*

(1) *$B = \{x \in P \colon x \mid q\}$ as sets. In particular q is the zero element* 0.

(2) *$x \geq y$ if and only if $x \mid y$.*

(3) *$DL = \{p_0^{s_0} \cdots p_{m-1}^{s_{m-1}} : s_i = 0$ or $e_i\}$.*

(4) *$N = \{x \in B \colon p_0 \cdots p_{m-1} \mid x$ in $P\}$.*

(5) *$B = \prod_{i=0}^{n-1} \beta(P/p_i^{e_i} P)$.*

Proof. For an element $x \in P$, we consider an integer $e_i(x)$ and an element $x_{(q)}$ defined by

$$e_i(x) := \max\{e\colon e \leq e_i \text{ and } p_i^e \mid x\}, \quad \text{and} \quad x_{(q)} := \textstyle\prod_{0 \leq i < m} p_i^{e_i(x)}.$$

We see that

$$x = x_{(q)} \in \beta(P/qP) \quad \text{for any } x \in P. \tag{4.4.2}$$

Indeed, $x_{(q)}$ divides x, and so $x \leq x_{(q)}$. If $xy = 0$ in P/qP, then xy is divisible by q in P. Therefore, $q \mid x_{(q)}y_{(q)}$ and so $q \mid x_{(q)}y$. Hence $x_{(q)}y = 0$ in P/qP and so $x_{(q)} \leq x$.

The statements (1)-(4) follow immediately from (4.4.2), and (5) from (1). □

Corollary 4.4.3. *We have isomorphisms of monoidal posets*

$$\beta(P/p_0^{e_0} \cdots p_{n-1}^{e_{n-1}} P) = \textstyle\prod_{i=0}^{n-1} \beta(\mathbb{Z}/2^{e_i}\mathbb{Z}) \quad \textit{and}$$
$$DL(\beta(P/p_0^{e_0} \cdots p_{n-1}^{e_{n-1}} P)) = \textstyle\prod_{i=0}^{n-1} \mathbb{Z}/2.$$

Corollary 4.4.4. *For any PID P and a non-zero element $q \in P$, the Bousfield lattice $\beta(P/qP)$ is monoidally distributive.*

Proof. Noticing the relation

$$(p_0^{s_0} \cdots p_{n-1}^{s_{n-1}}) \vee (p_0^{t_0} \cdots p_{n-1}^{t_{n-1}}) = p_0^{l_0} \cdots p_{n-1}^{l_{n-1}} \text{ with } l_i = \min\{s_i, t_i\},$$

the proof is straightforward. □

Theorem 4.4.5. *If P is a PID and $q \in P \setminus \{0\}$, then GRC2 holds on $\beta(P/qP)$, and so does GRC1.*

Proof. The ideal $N(\beta(P/qP))$ has the greatest element $g = p_0 \cdots p_{n-1}$. We compute

$$\begin{aligned}(p_0^{s_0} \cdots p_{n-1}^{s_{n-1}}) \vee g &= p_0^{\min\{s_0,1\}} \cdots p_{n-1}^{\min\{s_{n-1},1\}} = p_0^{\min\{2s_0,1\}} \cdots p_{n-1}^{\min\{2s_{n-1},1\}} \\ &= (p_0^{2s_0} \cdots p_{n-1}^{2s_{n-1}}) \vee g = (p_0^{s_0} \cdots p_{n-1}^{s_{n-1}})^2 \vee g.\end{aligned}$$

So the theorem follows from Theorem 4.3.22. □

Remark 4.4.6. We have another proof of the theorem. Since $\beta(P/qP) = \prod_{i=0}^{n-1} \beta(\mathbb{Z}/2^{e_i}\mathbb{Z})$ and GRC2 holds on $\beta(\mathbb{Z}/2^{e_i}\mathbb{Z})$, GRC2 holds on $\beta(P/qP)$ by Proposition 4.3.25.

4.5 Bousfield lattices of stable homotopy categories

Let Λ_E for a spectrum E denote the stable homotopy category of E-local spectra, and $\mathbb{B}(\Lambda_E)$ the Bousfield lattice in the sense of Bousfield. Then we have the Bousfield localization functor $L_E \colon \mathcal{S} \to \mathcal{L}_E$. The monoidal structure of $\mathcal{L}_E$ is given by $XY = L_E(X \wedge Y)$. We consider the Johnson-Wilson spectra $E(n)$ and the Morava K-theories $K(n)$ for $n \geq 0$. By the chromatic viewpoint, investigating the categories $\Lambda_n (= \Lambda_{E(n)})$ and $\Lambda_{K(n)}$ is one of main targets of stable homotopy theory. We determine the Bousfield lattices of these categories.

We begin with a simple category. A spectrum F is called a *field* if it is a ring spectrum and $F \wedge X = \bigvee \Sigma^a F$ for all spectra X.

Proposition 4.5.1. *Let F be a field. Then, $\mathbb{B}(\Lambda_F) = \mathbb{Z}/2$.*

Proof. Since F is a ring spectrum, we have $FX = F \wedge X$. We see easily $\langle X \rangle \geq \langle FX \rangle$. Suppose that $(FX)C = 0$. Then, XC is F-acyclic and so $XC = 0$. It follows that $\langle X \rangle = \langle FX \rangle = \langle \bigvee \Sigma^i F \rangle = 0$ or $\langle F \rangle$, which shows the lemma. □

By [4], the Eilenberg-MacLane spectrum $H\mathbb{Z}/p$ and the Morava K-theories $K(n)$ are fields.

Corollary 4.5.2. $\mathbb{B}(\Lambda_{H\mathbb{Z}/p}) = \mathbb{Z}/2 = \mathbb{B}(\Lambda_{K(n)})$.

Theorem 4.5.3. *Let $p_0, \ldots, p_n$ be $n+1$ distinguished prime numbers. Then $\mathbb{B}(\Lambda_n)$ is isomorphic to $\beta(\mathbb{Z}/p_0 \cdots p_n) = \prod_{i=0}^{n} \mathbb{Z}/2$ in $\mathcal{MP}$.*

Proof. The Bousfield lattice $\mathbb{B}(\mathcal{L}_n)$ consists of $\langle L_n X \rangle$ for all spectra X, which equals, by Ravenel [9],

$$\begin{aligned}\langle L_n X \rangle &= \langle L_n S^0 \rangle \cdot \langle X \rangle = \langle E(n) \rangle \cdot \langle X \rangle \\ &= \Big(\bigvee_{0 \leq i \leq n} \langle K(i) \rangle \Big) \cdot \langle X \rangle = \bigvee_{0 \leq i \leq n \textit{ and } K(i) \wedge X \neq 0} \langle K(i) \rangle.\end{aligned}$$

since L_n is smashing and $K(n)$ is a field. Here $\langle X \rangle \cdot \langle Y \rangle$ is the Bousfield class of the smash product $X \wedge Y$. We define a map $f \colon \mathbb{B}(\mathcal{L}_n) \to \beta(\mathbb{Z}/p_0 \cdots p_n)$ by

$f(\bigvee_{i\in S}\langle K(i)\rangle) = \prod_{i\notin S} p_i$ for $S \subset \{0,1,\cdots,n\}$. Then f preserves multiplication, since

$$\begin{aligned}(\textstyle\bigvee_{i\in S}\langle K(i)\rangle)(\bigvee_{j\in T}\langle K(j)\rangle) &= \textstyle\bigvee_{i\in S\cap T}\langle K(i)\rangle,\\ (\textstyle\prod_{i\notin S} p_i)(\prod_{j\notin T} p_j) &= \textstyle\prod_{i\notin S \ or \ i\notin T} p_i \ = \ \prod_{i\notin S\cap T} p_i.\end{aligned}$$

Moreover, for the order, we have

$$\begin{aligned}\textstyle\bigvee_{i\in S}\langle K(i)\rangle \leq \bigvee_{i\in T}\langle K(i)\rangle &\Leftrightarrow S\subset T \Leftrightarrow I(n)-S \supset I(n)-T\\ &\Leftrightarrow \textstyle\prod_{i\notin S} p_i \leq \prod_{i\notin T} p_i,\end{aligned}$$

and f is a monoidal poset map. □

A similar argument shows the following

Theorem 4.5.4. *Let $E = \bigvee_{i\in F} K(i)$ be a spectrum for a finite subset F of $\mathbb{Z}_{\geq 0}$. Then $\mathbb{B}(\mathcal{L}_E)$ is isomorphic to $\prod_{i\in F}\mathbb{Z}/2$.*

This together with Theorem 4.4.5 implies

Corollary 4.5.5. *GRC2 holds on $\mathbb{B}(\Lambda_E)$ for a spectrum $E = \bigvee_{i\in F} K(i)$ on a finite subset F of $\mathbb{Z}_{\geq 0}$.*

The chromatic tower $\Lambda_0 \leftarrow \Lambda_1 \leftarrow \Lambda_2 \leftarrow \cdots$ induces the inverse system

$$\mathbb{B}(\Lambda_0) \leftarrow \mathbb{B}(\Lambda_1) \leftarrow \mathbb{B}(\Lambda_2) \leftarrow \cdots . \tag{4.5.6}$$

Moreover, we notice that $B_\infty := \lim_n \mathbb{B}(\Lambda_n) = \prod_n \mathbb{Z}/2$ in $\mathcal{MP}$. We call a spectrum *harmonic* if it is $(\bigvee_{i\geq 0} K(i))$-local.

Theorem 4.5.7. *Let $\mathcal{H}$ be the stable homotopy category of harmonic spectra. Then $\mathbb{B}(\mathcal{H})$ is isomorphic to B_∞ in $\mathcal{MP}$.*

Proof. Let $f\colon \prod\mathbb{Z}/2 \to \mathbb{B}(\mathcal{H})$ be the poset map defined by $(x_n)\mapsto \bigvee_{x_n=1}\langle K(n)\rangle$ and let $p_n\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\Lambda_n)$ be the poset map defined by $\langle X\rangle \mapsto \langle X\rangle\cdot\langle E(n)\rangle$. Then, we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{B}(\Lambda_i) & \mathbb{B}(\Lambda_j) \arrow[l] \\
\mathbb{B}(\mathcal{H}) \arrow[u, "p_i"] \arrow[ur, "p_j"] & \prod\mathbb{Z}/2 \arrow[l, "f"'] \arrow[u] \arrow[ul]
\end{tikzcd}$$

for any i and j with $i \leq j$, since

$$\begin{aligned}p_i f((x_n)) &= p_i(\textstyle\bigvee_{x_n=1}\langle K(n)\rangle) \ = \ \bigvee_{x_n=1}\langle K(n)\rangle\cdot\langle E(i)\rangle\\ &= \textstyle\bigvee_{i\geq n,\ x_n=1}\langle K(n)\rangle.\end{aligned}$$

Therefore, $\mathbb{B}(\mathcal{H})$ is the inverse limit of the above system (4.5.6) by definition. □

Proof of Theorem 4.1.2. This follows from Theorem 4.5.7 and Proposition 4.3.25. □

In the same way, we obtain

Theorem 4.5.8. *Let T be a set of field spectra, and put $\bigvee T = \bigvee_{F \in T} F$. Then, $\mathbb{B}(\mathcal{L}_{\bigvee T}) = \prod \mathbb{Z}/2$.*

4.6 Problems

We leave some problems in this section.

Problem 4.6.1. *What is a condition on $X \xrightarrow{f} Y$ in $\mathcal{M}$, under which $\beta(f)$ is an isomorphism ?*

Suppose that the problem is settled and we find a map from $\mathbb{B}$ to a commutative monoid Y such that $\beta(f)$ is an isomorphism. Then, we may study $\mathbb{B} = \beta(\mathbb{B})$ by observing $\beta(Y)$ by virtue of Proposition 4.2.16, which may let us consider the lattice from a different viewpoint.

Problem 4.6.2. *Let M be a monoid with 0. Then, is there a ring R such that $\beta(M)$ is isomorphic to R as a monoid ?*

Example 4.6.3. *Let $p_0, \dots, p_n$ be $n+1$ distinguished primes. Then $\beta(\mathbb{Z}/p_0 \dots p_n) = \prod_{i=0}^{n} \mathbb{Z}/2$ as monoids by Theorem 4.5.3.*

If this is possible, we may approach these from the ring theoretic viewpoint.

Problem 4.6.4. *Are $\mathbb{B}/J(h)$ and $\mathbb{DL}$ monoidal posets?*

Bibliography

[1] A. K. Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. **54** (1979), 368-377.

[2] W. G. Dwyer and J. H. Palmieri, *Ohkawa's theorem: there is a set of Bousfield classes*. Proc. Amer. Math. Soc., **129** (2001), 881–886.

[3] W. G. Dwyer and J. H. Palmieri, *The Bousfield lattice for truncated polynomial algebras*, Homology, Homotopy and Applications, **10** (2008), 413-436.

[4] M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. Math. **148** (1998), 1-49.

[5] M. Hovey and J. H. Palmieri, *The structure of the Bousfield lattice*, Contemp. Math. **239** (1999), 175-196.

[6] M. Hovey, J. H. Palmieri and N. P. Strickland, *Axiomatic stable homotopy theory*, Mem. Am. Math. Soc. **610** (1997).

[7] S. B. Iyengar and H. Krause, *The Bousfield lattice of a triangulated category and stratification*, Math. Z. **273** (2013), 1215-1241.

[8] T. Ohkawa, *The injective hull of homotopy types with respect to generalized homology functors*, Hiroshima Math. J. **19** (1989), 631-639.

[9] D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. , 106 (1984), 351-414.