On aperiodic tilings by the projection method

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In 1982 quasi-crystals with icosahedral symmetry were discovered. (published in 1984). It had been axiomatic that the structure of a crystal was periodic, like a wallpaper pattern. Periodicity is another name for translational symmetry. Icosahedral symmetry is incompatible with translational symmetry and therefore quasi-crystals are not periodic. Most famous 2-dimensional mathematical model for a quasi-crystal is a Penrose tiling of the plane. In 1981 de Bruijn introduced projection methods to construct aperiodic tilings such as Penrose tilings.

We recall the definition of tilings by the projection method.

$L$: a lattice in $\mathbb{R}^d$ with a basis $\{b_i| i = 1, 2, \cdots , d\}$.

$E$: a $p$-dimensional subspace of $\mathbb{R}^d$,

$E^\perp$: its orthogonal complement.

$\pi: \mathbb{R}^d \rightarrow E$, $\pi^\perp: \mathbb{R}^d \rightarrow E^\perp$: the orthogonal projections.

$A$: a Voronoi cell of $L$

For any $x \in \mathbb{R}^d$ we put

$$W_x = \pi^\perp(x) + \pi^\perp(A) = \{\pi^\perp(x) + u | u \in \pi^\perp(A)\}$$

$$\Lambda(x) = \pi((W_x \times E) \cap L).$$

The Voronoi cell of a point $v \in \Lambda(x)$

$$V(v) = \{u \in \mathbb{R}^n||v - u| \leq |y - u|, \text{for all } y \in \Lambda(x)\}.$$

$V(x)$: the Voronoi tiling induced by $\Lambda(x)$, which consists of the Voronoi cells of $\Lambda(x)$.

For a vertex $v$ in $V(x)$

$$S(v) = \bigcup\{P \in V(x)|v \in P\}.$$

The tiling $T(x)$ given by the projection method is defined as the collection of tiles $\text{Conv} (S(v) \cap \Lambda(x))$, where $\text{Conv} (B)$ denotes the convex hull of a set $B$. Note that $\Lambda(x)$ is the set of the vertices of $T(x)$. 


In order to state theorems we recall several definitions. The dual lattice \( L^* \) is defined by the set of vectors \( y \in \mathbb{R}^d \) such that \( \langle y, x \rangle \in \mathbb{Z} \) for all \( x \in L \), where \( \langle \ , \ \rangle \) denotes standard inner product. A lattice \( L \) is called integral if all its vectors satisfy that \( \langle x, y \rangle \in \mathbb{Z} \) for all \( x, y \in L \). The standard lattice is both integral and self dual.

For \( L = \mathbb{Z}^d \), C. Hillman characterized the number of periods of the tilings. He also constructed periods for given tilings.

One of Hillman’s results is extended to the case that \( L \) is integral.

Theorem. Let \( T(x) \) be the tiling by the projection method and assume that \( L \) is integral. Then, \( \text{rank} \ \ker (\pi^{|L}|) \) is equal to the dimension of the linear space of the periods of \( T(x) \).

For the general lattices Theorem is not true. We have the following example;
\( L \) : a lattice in \( \mathbb{R}^2 \) with a basis \( \{(1, \sqrt{2}), (1, -1)\} \),
\( E \) : the \( x \)-axis of \( \mathbb{R}^2 \).

In this case it is easy to see that all tilings in \( \mathbb{R}^1 \) obtained by the projection method are periodic and \( \text{rank} \ \ker (\pi^{|L}|) = 0 \).

The following property is analogous to classical uniform distribution of sequences.

Theorem (de Bruijn and Senechal, 1995)
Assume that \( \pi^{|L}| \) is dense in \( E^\perp \).
\( K_1, K_2 : (d - p) \)-dimensional cubes in \( E^\perp \)
\( J \subset E : a \ p \)-dimensional cube centered at the origin.
For any positive real number \( \lambda \), we set
\[ P_1^\lambda = K_1 \times \lambda J, \quad P_2^\lambda = K_2 \times \lambda J. \]

Then,
\[ \lim_{\lambda \to \infty} \frac{\text{card} \ P_1^\lambda \cap L}{\text{card} \ P_2^\lambda \cap L} = \frac{\text{Vol}(K_1)}{\text{Vol}(K_2)}. \]
A tiling space \( T(E) \) is defined by a space of tilings consisting of all translates by \( E = \mathbb{R}^p \) of the tilings \( T(x) \) for all \( x \in E^\perp \). Tiling spaces are topological dynamical systems, with a continuous \( \mathbb{R}^p \) translation action and a topology defined by a tiling metric on tilings of \( \mathbb{R}^p \).

Let \( \text{Orb}(T(x)) \) denote the orbit of \( T(x) \) in \( T(E) \) by the \( \mathbb{R}^p \) translation action and \( \text{span}(A) \) denote the \( \mathbb{R} \)-linear span of a set \( A \).

Uniform distribution of the projection method is closely related to the ergodicity of the tiling space.

**Theorem**

Let \( T(E) \) be the tiling by the projection method in terms of a \( p \)-dimensional subspace \( E \) of \( \mathbb{R}^d \) and \( p' : E^\perp \to \text{span}(L^* \cap E^\perp) \) be the orthogonal projection. Define \( p : L \to \text{span}(L^* \cap E^\perp) \) by \( p = p' \circ (\pi|_L) \). We take a basis \( x_1, \ldots, x_k \) of the direct summand \( K \) such that \( L = p^{-1}(\{0\}) \oplus K \). Then \( T(E) \) decomposes into a \( k \) parameter family of orbit closures \( \text{Orb}(T(t_1x_1 + \cdots + t_kx_k)) \) for \( t_1, \ldots, t_k \in \mathbb{R} \).

In particular, we obtain that \( k \) is equal to \( \text{rank } (L^* \cap E^\perp) \).

Note that \( \pi^\perp(L) \) is dense in \( E^\perp \) if and only if \( E^\perp \cap L^* = \{0\} \). A. Hof(1988) proved that \( E^\perp \cap L^* = \{0\} \) if and only if \( T(E) = \overline{\text{Orb}(T(0))} \). Assume that \( L \) is integral. Then we see that \( \text{rank } (L^* \cap E^\perp) = \text{rank } (L \cap E^\perp) = \text{rank } \ker (\pi|_L) \) because \( L \subset L^* \) and \( L^*/L \) is finite. The number of independent periods of the tiling space \( T(E^\perp) \) is equal to \( \text{rank } \ker (\pi|_L) \).

We immediately obtain the following theorem in the case that \( L \) is integral:

**Theorem**

Let \( T(E) \) (resp. \( T(E^\perp) \)) be the tiling space by the projection method in terms of a \( p \)-dimensional subspace \( E \) (resp. \( (d-p) \)-dimensional subspace \( E^\perp \)) of \( \mathbb{R}^d \) and assume that \( L \) is an integral lattice. Then \( T(E) \) decomposes into a \( k \) parameter family of orbit closures, where \( k \) is equal to the number of independent periods of the tiling space \( T(E^\perp) \).