Addendum to the paper: A Dual to the Equivariant Bordism

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Let $G$ be a compact Lie group and $X$ a $G$-space. A $G$-drobism element of $X$ is a triple $(M, f, \phi)$ such that $M$ is a closed $G$-manifold with a smooth $G$-action $\phi : G \times M \to M$ and $f : X \to M$ is a $G$-map. Two $G$-drobism elements $(M, f, \phi)$ and $(M', f', \phi')$ are equivalent if there is a triple $(Q, F, \phi)$ such that

(1) $Q$ is a compact smooth $G$-manifold with boundary, the boundary of $Q$, $\partial Q$ being the disjoint union of $M$ and $M'$.

(2) $F : X \times I \to Q$ is a $G$-map with $F \mid X \times 0 = f$, $F \mid X \times 1 = f'$. 

(3) $\phi : G \times Q \to Q$ is a smooth $G$-action with $\phi \mid G \times M = \phi$, $\phi \mid G \times M' = \phi'$. 

The set of equivalence classes of $G$-drobism elements of a $G$-space $X$ is called the $G$-drobism set of $G$-space $X$ and will be denoted $(X, N^\sigma_\theta)$. Under the product operation; $(M, f, \phi) \cdot (N, h, \lambda) = (M \times N, (f \times h) \Delta, \phi \times \lambda)$, where $\Delta : X \to X \times X$ is the diagonal map, the set $(X, N^\sigma_\theta)$ forms a semi-group. Let $m_\sigma(X)$ be a group associated with the semi-group $(X, N^\sigma_\theta)$.

We proved the following theorems ([2]).

Theorem 1 ([2]).

The contravariant functor $m_\sigma(-)$ defines an equivariant cohomology theory on the category of pairs with a $G$-action to the category of abelian groups ([1]).

Let $G$ be a compact Lie group, $X$ a $G$-space. If $(M, f, \phi) \in (X, N^\sigma_\theta)$, let $\lambda (M, f, \phi) \in KO_\sigma (X)$ be the class of the $G$-bundle $f^*\tau_M$, where $\tau_M$ denotes the tangent bundle on $M$. This map $\lambda$ defines a homomorphism $\lambda : m_\sigma(X) \to KO_\sigma(X)$.

Theorem 2 ([2]).

If $X$ is a compact closed $G$-manifold, then the natural transformation $\lambda$ is an isomorphism.

Definition 3.

Let $G$ be a compact Lie group and $Y$ a $G$-space. A relative $G$-complex $(X, Y)$ is a $G$-space $X$ obtained inductively as follows:

Let $\overline{X}^{i-1} = Y$. Define $\overline{X}^i$ to be the result of adjoining arbitrarily many $G$-cells of arbitrary type but dimension $i$ to $\overline{X}^{i-1}$; we give $\overline{X}^i$ the weak topology and the natural $G$-action.

Let $X = \cup \overline{X}^i$, with the weak topology. A $G$-complex $X$ is a relative $G$-complex $(X, \phi)$, a $G$-subcomplex $Y$ of a $G$-complex $X$ is a $G$-complex such that $(X, Y)$ is a relative $G$-complex.
Lemma 4 (Willson [3]).

Let $G$ be a compact Lie group, let $h^*_o$ and $k^*_o$ be $G$-cohomology theories. Suppose $\lambda : h^*_o \rightarrow k^*_o$ is a natural transformation. Suppose for each $n$ that $\lambda : h^n_o(pt) \rightarrow k^n_o(pt)$ is an isomorphism. Then $\lambda$ is a natural equivalence and $h^n_o(X, Y) = k^n_o(X, Y)$ for any finite $G$-complex $(X, Y)$.

Theorem 5.

Let $G$ be a compact Lie group and $X$ a finite $G$-complex. Then $\lambda : m_o(X) \rightarrow KO_o(X)$ is an isomorphism.

Proof. By Theorem 2 ((2)) for $X = pt$, the natural transformation $\lambda : m_o(pt) \rightarrow KO_o(pt)$ is an isomorphism, so Lemma 5 ((3)), $\lambda : m_o(X) \rightarrow KO_o(X)$ is an isomorphism.

Corollary 6 (Theorem 2).

Let $G$ be a compact Lie group, $X$ a compact smooth $G$-manifold. Then the natural transformation $\lambda : m_o(X) \rightarrow KO_o(X)$ is an isomorphism.

Proof. If $G$ is a compact Lie group, then all compact smooth $G$-manifolds admit the structure of a finite $G$-complex ((4)).

References


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